Say *aaaaaa*: Unary Graph Automatic Semigroups

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An alphabet is a set Σ of symbols, e.g. $\Sigma = \{a, b\}$.The elements of Σ are letters.

A word over our alphabet is a string of letters from Σ, e.g. *aaa*, *ababa*, *abbb*.The set of all possible words over an alphabet is denoted Σ^*. This contains the empty word ϵ.

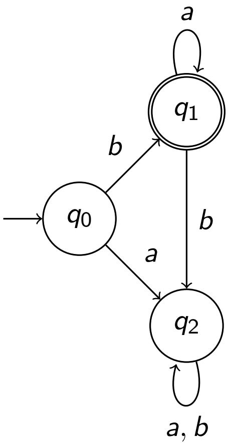
A language is any subset of Σ^*, e.g.
$\{a, b, ab, ba\}, \{ba^n : n \in \mathbb{N}\}, \{a^n b^n : n \in \mathbb{N}\}$.

Finite state automata

An automaton is a 5-tuple $(\Sigma, S, q_0, F, \delta)$, where

- Σ is a finite alphabet.
- S is a finite set of states.
- $q_0 \in S$ is a distinguished start state.
- $F \subseteq S$ is a set of final states.
- $\delta : S \times \Sigma \to S$ is a transition function.

An example



This automaton accepts the language $\{ba^n : n \in \mathbb{N}\}$.

Regular languages are precisely those which are accepted by some finite state automaton.

Two-tape automata

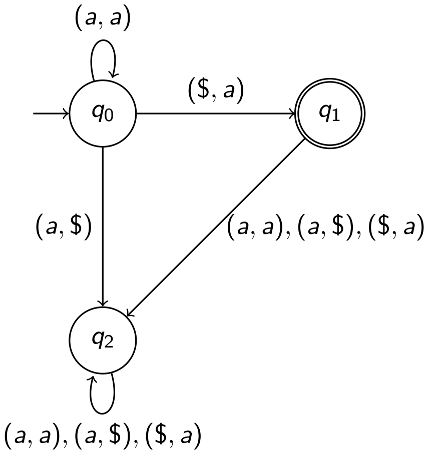
We may also consider automata that accept pairs of words $(\alpha, \beta) \in \Sigma^* \times \Sigma^*$.

If $\alpha = a_1 a_2 \ldots a_m$ and $\beta = b_1 b_2 \ldots b_n$ then we read pairs $(a_1, b_1), (a_2, b_2)$ etc. We introduce a padding symbol \$ to deal with the case where $m \neq n$.

This gives us a regular language over an alphabets of the form

$$(\Sigma \cup \{\$\})^2 - \{(\$, \$)\}.$$

An example



This automaton accepts the language $\{(a^n, a^{n+1}) : n \geq 0\}$.

Graph automatic semigroups

Let S be a semigroup generated by a finite set X. We call S *graph automatic* iff there exisits a finite alphabet Σ, a regular language $R \subseteq \Sigma^*$, and an onto map $\nu : R \to S$ such that the sets

$$R_= = \{(u, v) \in R \times R : \nu(u) = \nu(v)\}$$

and

$$R_x = \{(u, v) \in R \times R : \nu(u)x = \nu(v)\}$$

for $x \in X$ are regular.

We say that (X, Σ, R, ν) is a *graph automatic structure* for S.

If $\Sigma = X$ and ν is a homomorphism then we have an *automatic semigroup*.

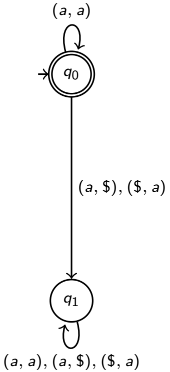
An example

Let $S = \mathbb{N}$ under addition with $X = \{1\}$. Then we may take:

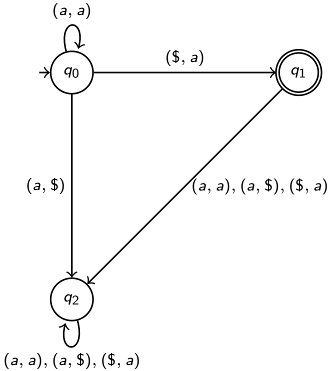
- $\Sigma = \{a\}$,
- $R = a^*$,
- $\nu : a^* \to \mathbb{N}$ defined by $\nu(a^n) = n$.

This is a graph automatic structure for $\mathbb{N}$.

Automaton accepting $R_=$:



Automaton accepting R_1:



FA-presentable structures

Let $\mathcal{A} = (A; P_0, \ldots, P_k)$ be a relational structure, where each P_i is a relation of arity p_i. Then $\mathcal{A}$ is *FA-presentable* if there is a regular language L over some finite alphabet Σ and a surjective mapping $\phi : L \to A$ such that

$$L_= = \{(\alpha, \beta) \in L \times L : \phi(\alpha) = \phi(\beta)\}$$

and

$$L_{P_i} = \{(\alpha_1, \ldots \alpha_{p_i}) \in L^{r_i} : (\phi(\alpha_1), \ldots, \phi(\alpha_{p_i})) \in P_i\}$$

are regular languages.

Graph automatic semigroups are those whose Cayley graphs have an automatic presentation.

If a structure is FA-presentable, then it has an injective automatic presentation (Khoussainov and Nerode).

If a structure is FA-presentable, then it has a binary automatic presentation (Blumensath).

So any graph automatic semigroup has an injective binary graph automatic structure.

Unary languages

A semigroup is *unary graph automatic* if it has a graph automatic structure over a one letter alphabet.

Does every graph automatic semigroup have a unary graph automatic structure?

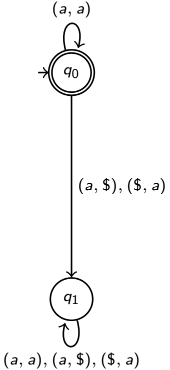
Every finite graph automatic semigroup is unary graph automatic.

If an infinite structure is unary FA-presentable, then in has an injective unary automatic presentation over the language a^* (Cain, Ruškuc, Thomas). So for infinite semigroups, we need only consider whether we have an injective unary graph automatic structure $(X, a, a^*, \nu : a^* \to S)$.

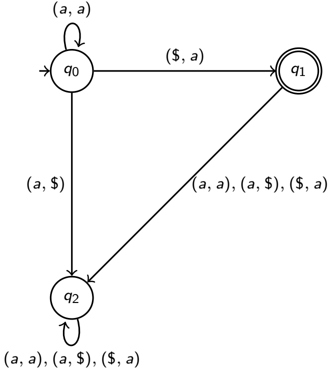
Unary automata

Consider the format of the acceptor automata for a unary graph automatic semigroup. These will be automata over the alphabet $\{(a, a), (a, \$), (\$, a)\}$.

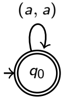
We assume that our automata are deterministic and we consider only those paths which lead to the accept states.



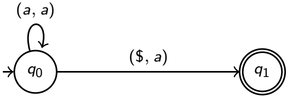
(a, a)
q_0
(a, \$), (\$, a)
q_1
(a, a), (a, \$), (\$, a)



(a, a)
q_0
(\$, a)
q_1
(a, \$)
(a, a), (a, \$), (\$, a)
q_2
(a, a), (a, \$), (\$, a)



(a, a)
q0



(a, a)
q0
($, a)
q1

Unary automata

Consider the format of the acceptor automata for a unary graph automatic semigroup. These will be automata over the alphabet $\{(a, a), (a, \$), (\$, a)\}$.

We assume that our automata are deterministic and we consider only those paths which lead to the accept states.

Note that we must have at least one circuit or loop, as our language is infinite, and if we read a \$ in a component then we must continue to read only \$ afterwards.

Our automata have the following restrictions:

- An acceptor automaton cannot have a $(\$, a)$ circuit.
- An acceptor automaton for a unary graph automatic semigroup cannot have two consecutive circuits.
- If an acceptor automaton for a unary graph automatic semigroup has two distinct circuits then these circuits must have the same length p. Moreover, each accept state will accept words of a different remainder when their lengths are considered modulo p.
- If an acceptor automaton for a graph automatic semigroup consists of only one circuit labelled $(a, \$)$ then this circuit has length one, i.e. is a loop.

This means that an acceptor automaton for a unary graph automatic semigroup has one of the following formats:

1. A finite path followed by a single loop of the form $(\$, a)$.
2. A finite path followed by a single circuit of the form (a, a), possibly with finite offshoots labelled $(a, \$)$ or $(\$, a)$.
3. A finite path followed by two circuits, one labelled (a, a) (possibly with finite offshoots) and one labelled $(\$, a)$, where both circuits have the same length q and for remainders $r_0, \ldots, r_{q-1}$ and some $0 < k < q-1$, we have that $\{a^{nq+r_0}, \ldots, a^{nq+r_k} : n \in \mathbb{N}_0\}$ are accepted by states on the $(\$, a)$ circuit and $\{a^{nq+r_{k+1}}, \ldots, a^{nq+r_{q-1}} : n \in \mathbb{N}_0\}$ are accepted by states on or offshoots of the (a, a) circuit.

Theorem

Let S be a unary graph automatic semigroup which is left-cancellative. Let $x \in S$ be of infinite order and right-cancellative. Then S can be written as Ax^* for some finite set $A \subseteq S$.

Sketch of proof

- Let $(X, a, a^* = R, \nu)$ be an injective unary graph automatic structure for S, with $x \in X$.
- Consider $\mathscr{A}_x$, the automaton accepting R_x. As x is right-cancellative, this must be of type 2.
- Use the accept states of $\mathscr{A}_x$ to partition a^* according to the first component. This gives us a finite set F plus finitely many arithmetic progressions.
- Repeatedly multiplying by x gives us a path through these arithmetic progressions, starting in a finite set and reaching any of our words a^n.
- Converting this back into the terms of our semigroup we get $S = Ax^*$.

Theorem

The semigroup $\mathbb{N}_0 \times \mathbb{N}_0$ is graph automatic.

Proof.

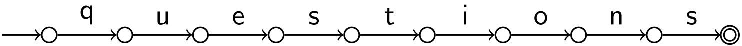
We have seen that $\mathbb{N}$ is unary graph automatic. Adding identities and taking direct products (if the product is finitely generated) both preserve graph automaticity. $\square$

Theorem

The semigroup $\mathbb{N}_0 \times \mathbb{N}_0$ is not unary graph automatic.

Proof.

Suppose $\mathbb{N}_0 \times \mathbb{N}_0$ is unary graph automatic. As $\mathbb{N}_0 \times \mathbb{N}_0$ is cancellative and $(1, 0)$ is an element of infinite order we can apply the earlier theorem to get that $\mathbb{N}_0 \times \mathbb{N}_0 = A(1, 0)^*$ for some finite set A, a contradicion. □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