The Tensor Product (via the Universal Property)

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PPS Talk
University of St. Andrews

October 16, 2015

Overview of the Presentation

The Tensor Product

The tensor product of modules is a construction that allows multilinear maps to be carried out in terms of linear maps.

The tensor product can be constructed in many ways, such as using the basis of free modules. However, the standard, more comprehensive, definition of the tensor product stems from category theory and the universal property.

The Tensor Product (via the Universal Property)

Definition

Let $\mathbf{R}$ be a commutative ring. Let E_1, E_2 and F be modules over $\mathbf{R}$.
Consider bilinear maps of the form

$$f : E_1 \times E_2 \to F.$$

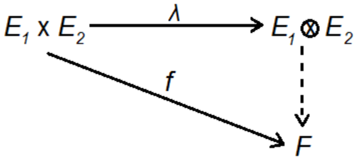
The Tensor Product (via the Universal Property)

Definition

The *tensor product* $E_1 \otimes E_2$ is the module with a bilinear map

$$\lambda \colon E_1 \times E_2 \to E_1 \otimes E_2$$

such that there exists a unique homomorphism which makes the following diagram commute.



A Construction of the Tensor Product

Here is a more constructive definition of the *tensor product*.

Definition

Let M be the free module generated by the set of all pairs $(x_1, x_2), x_i \in E_i$, i.e. generated by the set $E_1 \times E_2$.

Let N be the submodule of M generated by all elements fulfilling the following properties:

- $(x_1 + x_1', x_2) - (x_1, x_2) - (x_1', x_2)$
- $(x_1, x_2 + x_2') - (x_1, x_2) - (x_1, x_2')$
- $(ax_1, x_2) - a(x_1, x_2)$
- $(x_1, ax_2) - a(x_1, x_2)$

for all $x_i, x_i' \in E_i$ and $a \in \mathbf{R}$.

A Construction of the Tensor Product

Definition

Now consider the canonical injection $E_1 \times E_2 \to M$ and the canonical map $M \to M/N$. The composition of these maps

$$\phi \colon E_1 \times E_2 \to M/N$$

is bilinear and a tensor product.

A Construction of the Tensor Product

The module M/N is denoted by

$$E_1 \otimes E_2 \text{ or } \bigotimes_{i=1}^{2} E_i.$$

We have constructed a specific tensor product in the isomorphism class of tensor products, and we shall call it the *tensor product* of E_1 and E_2.

Properties of the Tensor Product

For $x_i \in E_i$, we can write

$$\phi(x_1, x_2) = x_1 \otimes x_2.$$

Then the following properties hold

- $(x_1 + x_1', x_2) = (x_1, x_2) + (x_1', x_2)$
- $(x_1, x_2 + x_2') = (x_1, x_2) + (x_1, x_2')$
- $(ax_1, x_2) = a(x_1, x_2)$
- $(x_1, ax_2) = a(x_1, x_2)$

for $x_i, x_i' \in E_i$ and $a \in \mathbf{R}$.

Properties of the Tensor Product

We note that every element of E_1 and E_2 can be written as a sum of terms $x \otimes y$ with $x \in E_1$ and $y \in E_2$, because such terms generate $E_1 \otimes E_2$, and

$$a(x \otimes y) = ax \otimes y = x \otimes ay$$

for $a \in \mathbf{R}$.

Example 1: Tensor Product over a Field

Consider a field F. Consider a vector space V over F with dimension m and basis $\{v_1, ..., v_m\}$, and a vector space W over F with dimension n and basis $\{w_1, ..., w_n\}$.

Then the tensor product $V \otimes W$ is the vector space spanned by the elements $v_i \otimes w_j$ $(1 \leq i \leq m, 1 \leq j \leq n)$ and has dimension mn.

Example 2: Collapsing Between Modules

The tensor product can involve a great deal of collapsing between modules.

Consider the tensor product over $\mathbb{Z}$ of $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}$, where m, n are integers > 1 and are relatively prime. Then the tensor product

$$\mathbb{Z}/n\mathbb{Z} \otimes \mathbb{Z}/m\mathbb{Z} = 0.$$

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