Time-changes of homogeneous flows

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Left-invariant flows

Consider a (connected, simply connected) Lie group G. Denote by L_g the left multiplication by $g \in G$, i.e $L_g : h \to gh$.

For any $\mathbf{w} \in \mathfrak{g}$ we can define a (left-invariant) vector field W on G by

$$W_g = (L_g)_* \mathbf{w}.$$

Indeed the map $\mathbf{w} \mapsto W$ is a bijection between $\mathfrak{g}$ and $\{$left-invariant vector fields on $G\}$.

The flow $\{\varphi_t^{\mathbf{w}}\}_{t \in \mathbb{R}}$ associated to W is explicitely given by

$$\varphi_t^{\mathbf{w}}(g) = g \cdot \exp(t\mathbf{w}).$$

A (maybe too easy) example

Consider $G = (\mathbb{R}^n, +)$. Then $\mathrm{Lie}(\mathbb{R}^n) \simeq \mathbb{R}^n$ and for $\mathbf{w} \in \mathbb{R}^n \setminus \{0\}$ everything boils down to

$$\varphi_t^{\mathbf{w}}(\mathbf{x}) = \mathbf{x} + t\mathbf{w}.$$

Every point eventually leaves any compact set: no recurrence, the dynamics is trivial.

So what? Choose your favourite lattice in $\mathbb{R}^n$, e.g. $\mathbb{Z}^n < \mathbb{R}^n$ and look at the same flow on the quotient space $\mathbb{R}^n/\mathbb{Z}^n \simeq \mathbb{T}^n$: linear flows on tori are definitely more interesting!

Homogeneous flows

Let Λ be a discrete subgroup of a Lie group G and let $M = \Lambda \backslash G$. A homogeneous flow is a flow on the manifold M given by a left-invariant vector field.

We will consider only lattices Λ, i.e. discrete subgroups such that the quotient M has finite left-Haar measure.

Proposition. If G contains a lattice, then it is unimodular.

In particular, our flow preserves the Haar measure —recall it is given by right-multiplication by $\exp(t\mathbf{w})$.

Geodesic and Horocycle flows

Let $G = \mathrm{PSL}_2(\mathbb{R})$; so

$$\mathfrak{g} = \{\mathbf{w} \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) : \mathrm{Tr}\,\mathbf{w} = 0\} = \langle \mathbf{x}, \mathbf{v}, \mathbf{u} \rangle,$$

where $\mathbf{x} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$, $\mathbf{v} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $\mathbf{u} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$.

The correspondent flows, which are given by right multiplication by $\exp(t\mathbf{x}) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$, $\exp(t\mathbf{v}) = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ and $\exp(t\mathbf{u}) = \left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)$ are called the Geodesic, (stable) Horocycle and (unstable) Horocycle flow respectively.

Geodesic and Horocycle flows II

A classical example is $\Lambda = \mathrm{PSL}_2(\mathbb{Z})$: the quotient $\Lambda\backslash G$ has finite volume but it is not compact. Indeed, it is isomorphic to the unit tangent bundle of the modular surface

$$\mathrm{PSL}_2(\mathbb{Z})\backslash \mathrm{PSL}_2(\mathbb{R}) \simeq T^1(\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})),$$

and $g \mapsto g \cdot \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$ is the geodesic flow induced by the hyperbolic metric on $\mathbb{H}$.

These flows have been studied intensively for years by very smart people and they have deep connections with number theory.

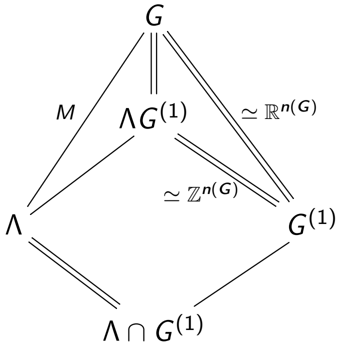
Nilflows

Let G be a n-step nilpotent Lie group, i.e. $\mathfrak{g}^{(n+1)} = \{0\}$ and $\mathfrak{g}^{(n)} \neq \{0\}$, where $\mathfrak{g}^{(1)} = \mathfrak{g}$ and $\mathfrak{g}^{(i)} = [\mathfrak{g}, \mathfrak{g}^{(i-1)}]$.
The manifold $M = \Lambda \backslash G$ is said to be a nilmanifold and the flow $\{\varphi_t^{\mathbf{w}}\}_{t \in \mathbb{R}}$ a nilflow.

Advantages:

- Λ is a lattice if and only if $\Lambda \backslash G$ is compact;
- $\exp \mathfrak{g} \to G$ is an analytic diffeomorphism;
- for almost every $\mathbf{w} \in \mathfrak{g}$ the corresponding nilflow is uniquely ergodic: every orbit equidistributes w.r.t. the Haar measure.

Ergodicity and mixing for nilflows



Ergodicity and mixing for nilflows II

We have an exact sequence

$$0 \to \Lambda \backslash \Lambda G^{(1)} \to M \xrightarrow{\pi} \mathbb{T}^{n(G)} \to 0,$$

so that the push-forward vector field $\pi_* W$ induces a linear flow on the torus $\mathbb{T}^{n(G)}$ —recall the first example.

Theorem. The flow induced by W on M is uniquely ergodic iff the flow induced by $\pi_* W$ on $\mathbb{T}^{n(G)}$ is ergodic (equivalently, uniquely ergodic).

However, these flows are not mixing.

Time-changes

Time-changes

A time-change of $\{\varphi_t\}_{t\in\mathbb{R}}$ is a flow with the same orbits as $\{\varphi_t\}_{t\in\mathbb{R}}$ but percorred at different times.

Formally, let $\alpha\colon M \to \mathbb{R}$ be smooth, the time-change associated to α is the flow $\{\varphi_t^\alpha\}_{t\in\mathbb{R}}$ induced by the vector field αW.

(Unique) Ergodicity is preserved by any positive time-change; on the contrary mixing is a delicate issue.

Some results

Theorem (Marcus - '77). Any sufficiently smooth time-change of the Horocycle flow on a compact surface is mixing.

Theorem (Forni, Ulcigrai - '12). "Quantitative" mixing + the spectrum of smooth time-changes of the Horocycle flow on compact surfaces is equivalent to Lebesgue.

Theorem (Avila, Forni, Ulcigrai - '11). Let H_1 be the Heisenberg group, i.e. the 3-dimensional 2-step nilpotent Lie group and consider a uniquely ergodic nilflow on H_1. Within a dense subspace, every nontrivial time-change is mixing.

My reasearch and other open questions

At the moment, I am trying to generalize the result by Avila, Forni and Ulcigrai to some classes of higher-dimensional and higher-step nilpotent groups.

Some open questions:

- Quantitative mixing for time-changes of Horocycle flows on noncompact finite-volume quotients?
- Mixing for time-changes of nilflows on generic nilpotent groups? And for other Lie groups?
- Quantitative mixing for time-changes of nilflows? Require some Diophantine condition on $\mathbf{w}$?
- ...