On Princesses and Decompositions

Some Aspects of Synchronization Theory

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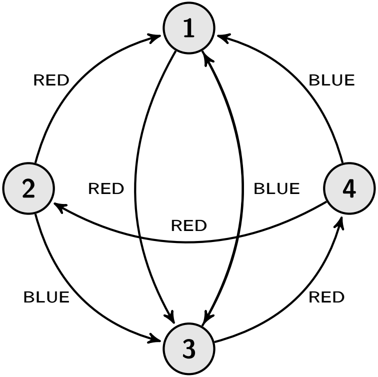
Synchronization



Follow BLUE, RED, BLUE, BLUE.

Definition

A semigroup S is synchronizing, if it contains a constant map. (S a finite transformation semigroup acting on a finite set X)

How does a constant map look like?

$$c = g_1 a_1 g_2 a_2 g_3 a_3 \dots g_k a_k g_{k+1} = a_1^{g_1} a_2^{g_1 g_2} a_3^{g_1 g_2 g_3} \dots a_k^{g_1 g_2 \cdots g_k} g_1 g_2 g_3 \dots g_k g_{k+1}.$$

(for $a^g = gag^{-1}$).

For $S = \langle G, a \rangle$, for some group G and a transformation a, this reduces to

$$c = a^{g_1} a^{g_1 g_2} \ldots a^{g_1 g_2 \cdots g_k} g_1 g_2 g_3 \ldots g_k g_{k+1}$$

c is constant if and only if c' is constant.

$$c' = a^{g_1} a^{g_1 g_2} \ldots a^{g_1 g_2 \cdots g_k}.$$

Hence, S is synchronizing, if and only if

$$\langle a^G \rangle \subseteq S$$

is synchronizing.

Normalizing Groups

Definition

A group G normalizes a transformation a (G is a-normalizing), if $\langle G, a \rangle = \langle a^G \rangle$.

Theorem: [Araujo, Cameron, Mitchell, Neunhöffer]

A group G normalizes all transformations $a \in T_n \setminus S_n$ if and only if G is one of

1. $\{1\}, A_n, S_n$, or
2. one of five other groups.

Now, consider $S = \langle G, a_1, a_2, ..., a_n \rangle$.

Definition

1. A group G is $(a_1, ..., a_n)$-**normalizing**, if $\langle G, a_i \rangle = \langle a_i^G \rangle$, for all i.
2. A group G is $\{a_1, ..., a_n\}$-normalizing, if $\langle G, a_1, ..., a_n \rangle = \langle a_1^G, ..., a_n^G \rangle$.
3. A group G is **strongly** $\{a_1, ..., a_n\}$-normalizing, if $\langle G, a_{j_1}, ..., a_{j_k} \rangle = \langle a_{j_1}^G, ..., a_{j_k}^G \rangle$, for any subset $\{a_{j_1}, ..., a_{j_k}\}$.

Properties: 3) $\Rightarrow$ 2) and 1), but what about other relations.

Disjoint Decompositions

Assume we can decompose S as follows

$$\langle G, a_1, ..., a_n \rangle \setminus G = \langle G, a_1 \rangle \setminus G \uplus \cdots \uplus \langle G, a_n \rangle \setminus G$$

Lemma

Given such a decomposition. Then, G is $(a_1, ..., a_r)$-normalizing implies G is $\{a_1, ...a_r\}$-normalizing.

Definition

$T = \{a_1, ..., a_r\}$, $r \geq 2$.

1. G is T-decomposing if the above decomposition holds.
2. G is **strongly** T-decomposing if for all subsets T' of T, G is T'-decomposing.

Strong Decomposition

Again, $T = \{a_1, ..., a_r\}$, $S = \langle G, T \rangle \setminus G$ and $S_1 = \langle G, a_1 \rangle \setminus G$. Then

$$S - S_1$$

is also a semigroup. -> "Differences" and "Sums" of semigroups are semigroups

For semigroups which are strongly decomposable $(a_1, ..., a_r)$-normalizing and strongly $\{a_1, ..., a_r\}$-normalizing is the same. Hence, this gives a close relation to semigroups of the form

$$\langle a_1^G, a_2^G, ..., a_r^G \rangle.$$

Most importantly: The effort of analysing $\langle G, a_1, ..., a_r \rangle$ is reduced to an analysis of $\langle G, a_i \rangle$, for all i.

So far ...

By construction, in the framework of this strong decomposition holds:

$$(a_1, ..., a_r)\text{-normalizing} \;\Leftrightarrow \text{strongly } \{a_1, ..., a_r\}\text{-normalizing} .$$

And both imply $\{a_1, ..., a_r\}$-normalizing .

Simplified properties:

1. check "regularity";
2. check "idempotent generated";
3. check minimal generating sets;
4. check automorphism groups (for some cases).

To Do:

1. Derive $L-$, $R-$, $D-$classes of S from its compontents;
2. Analysis of subsemigroups
3. ...

Which semigroups are (strongly) decomposable? And what is the difference?

Lemma

If a semigroup is decomposable (like above), then it is not synchronizing.

The best known examples of non-synchronizing semigroups come from endomorphism monoids of graphs.

Theorem [Cameron]

$\langle G, a \rangle$ is non-synchronizing, if and only if it contains endomorphisms of some graph (non-trivial with complete core).

Examples

Hamming Graphs: The endomorphism monoid of a Hamming graph consists of either bijections or Latin squares and Latin hypercubes.

1st case: 2 dimensions $\rightarrow$ the endomorphisms are Latin squares.

- $S' = S \setminus G$ is a simple semigroup.
- S' is strongly decomposable $\rightarrow$ we can find easily a minimal generating set T $(S' = \langle G, T\rangle \setminus G)$.
- If $S' = S_1 \uplus S_2$, then for $s_1 \in S_1$ and $s_2 \in S_2$ holds $s_1 s_2 \in S_1$ and $s_2 s_1 \in S_2$.

$\rightarrow$ ∞-family of examples for the strong decomposition

2nd case: m dimensions consider the minimal ideal. $\rightarrow$ same as in dimension 2, but with Latin hypercubes.

3rd case: m dimensions, consider the whole endomorphism monoid (non-bijective part) S'. $\rightarrow$ We can always find a (non-strong) decomposition.

Example

$S' = Sing(H(3,4))$; S' contains transformations of ranks 4 and 16 only. -> We can find a decomposition into 5 parts which is not strong.

$$S' = S_1 \uplus S_2 \uplus S_3 \uplus S_4 \uplus S_5.$$

Here, surprisingly holds: $s_1 s_2 \in S_3$, for some $s_1 \in S_1$ and $s_2 \in S_2$.

More Examples

- **Orthogonal array graphs** (Latin square graphs). Their endomorphisms are Latin squares.
- The **triangular graph** is the graph where the vertices are the subsets of size two of $\{1, ..., n\}$. Two vertices are adjacent if the sets intersect. Its endomorphisms are Latin squares.

We determined all (primitive) graphs ≤ 64 vertices which have proper endomorphisms (106 graphs). And we know the monoids of the smallest graphs (all up to 49 vertices). $\rightarrow$**All of the have the decomposition property** (or are simply generated).

Thank You for Your Attention