Diagram Semigroups

An adventure from permutations all the way to PBRs

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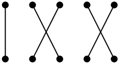
Permutations

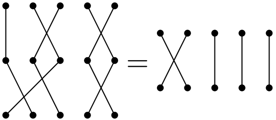
S_n – the symmetric group – the set of all permutations on $\mathbf{n} = \{1 \ldots n\}$ together with the operation of concatenation.

Permutation – a bijective function $\sigma : X \to X\{1 \ldots n\} \to \{1 \ldots n\}\mathbf{n} \to \mathbf{n}$.

A permutation can be written:

- in two-row notation, $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{pmatrix}$
- in disjoint cycle notation, $(2\ 3)(4\ 5)$
- as a diagram, 

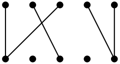
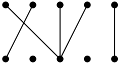
Multiplication by concatenating diagrams:



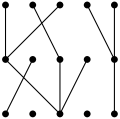
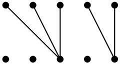
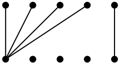
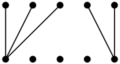
Transformations

Transformation – **any** function $\sigma : \mathbf{n} \to \mathbf{n}$.

T_n – the full transformation monoid

$\alpha =$ , $\beta =$ 

Calculate:

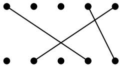
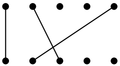
$\alpha\beta =$  $=$ , $\beta\alpha =$ , $\alpha^2 =$ 

$$\ker(\alpha) = \{\{1,3\},\{2\},\{4,5\}\},\ \mathrm{im}(\alpha) = \{1,3,5\},$$

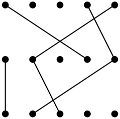
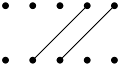
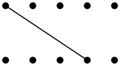
$$\ker(\beta) = \{\{1,3,4\},\{2\},\{5\}\},\ \mathrm{im}(\beta) = \{1,3,5\}.$$

Partial permutation – a bijective function $\sigma : X \to Y$, where $X, Y \subseteq \mathbf{n}$.

I_n – the symmetric inverse monoid

$\alpha =$  , $\beta =$ 

Calculate:

$\alpha\beta =$  $=$  , $\beta\alpha =$ 

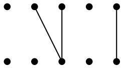
$$\mathrm{dom}(\alpha) = \{1, 4, 5\}, \qquad \mathrm{codom}(\alpha) = \{2, 4, 5\},$$

$$\mathrm{dom}(\beta) = \{1, 2, 5\}, \qquad \mathrm{codom}(\beta) = \{1, 2, 3\}.$$

Partial transformations

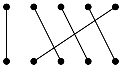
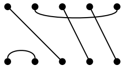
Partial transformation – **any** function $\sigma : X \to Y$, where $X, Y \subseteq \mathbf{n}$.

$$PT_n$$

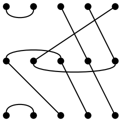
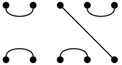
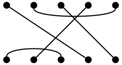


Brauer diagrams

$\mathfrak{B}_n$ – the Brauer monoid

$\alpha =$  , $\beta =$ 

Brauer diagram – any partition of $\mathbf{n} \cup \mathbf{n}'$ into pairs.

$\alpha\beta =$  $=$  , $\beta\alpha =$ 

$$\mathrm{dom}(\alpha) = \{3, 4, 5\}, \qquad \mathrm{codom}(\alpha) = \{2, 4, 5\},$$

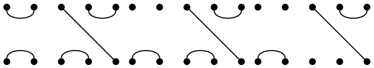
$$\ker(\alpha) = \{\{1, 2\}, \{3\}, \{4\}, \{5\}\}, \qquad \mathrm{coker}(\alpha) = \{\{1, 3\}, \{2\}, \{4\}, \{5\}\},$$

$$\mathrm{rank}(\alpha) = 3, \qquad \mathrm{rank}(\beta) = 3, \qquad \mathrm{rank}(\alpha\beta) = 1.$$

Partial Brauer diagrams

Partial Brauer diagram – any partition of $\mathbf{n} \cup \mathbf{n}'$ into pairssets of size up to 2.

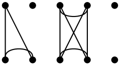
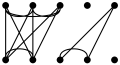
$$P\mathfrak{B}_n$$

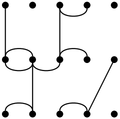
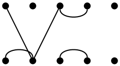
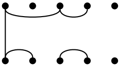


Bipartitions

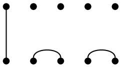
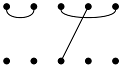
$\mathcal{P}_n$ – the partition monoid

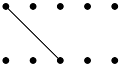
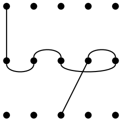
Bipartition – any equivalence relation on $\mathbf{n} \cup \mathbf{n}'$.

$\alpha =$  , $\beta =$ 

$\alpha\beta =$  $=$  $=$  , $\beta\alpha = \beta.$

Bipartitions

$\gamma =$ , $\delta =$ 

$\gamma\delta =$  $=$ 

Thank you for listening

Main source:

James East, Attila Egri-Nagy, Andrew R. Francis, James D. Mitchell, *Finite Diagram Semigroups: Extending the Computational Horizon*, https://arxiv.org/abs/1502.07150