DECISION PROBLEMS FOR FINITELY PRESENTED AND ONE-RELATION SEMIGROUPS AND MONOIDS

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ABSTRACT

For some distinguished properties of semigroups, such as having idempotents, regularity, hopficity etc, we study the following: whether that property or its negation is a Markov property, whether it is decidable for finitely presented semigroups and for one-relation semigroups and monoids. All the results and open problems are summarized in a table.

Keywords: Decision problem; semigroup presentation; Markov property.
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1 INTRODUCTION

At the beginning of 20th century Max Dehn posed the question whether every one-relator group has soluble word problem. Later this was answered in the affirmative by Magnus [Mag32]. At that stage it was natural to find an example of algorithmically insoluble problem. Based upon the results of Turing and Post it was possible to find the first undecidability result of an algebraic nature: Markov [Mar47] and Post [Pos47] established examples of finitely presented semigroups with insoluble word problem. This result was used later for proving other undecidability results like insolubility of the word problem for finitely presented groups [AD00]. Markov also used the word problem for finitely presented semigroups to prove that the so-called *Markov properties* are undecidable for finitely presented semigroups, see Sec. 3. Even though there has been much success to deal with algorithmic problems using semigroup theory, the original question of Dehn for one-relation semigroups still remains open. A major step in the approach to this problem uses combinatorics on words [Adj66, AO87, Lal74, Wat96]. An interesting result is obtained in [IMM01] where the word problem for one-relation semigroups is reduced to the word problem for one-relation inverse semigroups, the latter being closer to the class of groups. Some partial results, using the geometric approach of the so-called *word diagrams*, were obtained in [Rem80]. The least known number of relations in a finitely presented semigroup one needs to take to obtain a semigroup with insoluble word problem is 3, due to Matiyasevich [Mat95].

As a natural continuation, there appeared in literature some works on other algorithmic problems for finitely presented and one-relation semigroups, with greater emphasis on the latter. It was proved by Adian [Adj66] that cancellativity is decidable for one-relation semigroups. Lallement [Lal74] showed that it is decidable whether a one-relation semigroup has idempotents (we reprove this result in another fashion in Sec. 7). Zhang [Zha91, Zha92] proved partial results for the conjugacy problem for one-relation monoids.

In this paper we do the following. We construct a list of 'distinguished' properties for semigroups. For each property from this list we ask whether it or its negation is a Markov property. If so then this property is undecidable for finitely presented semigroups. If neither the property nor its negation is a Markov property (or we do not know the answer to these) then we use a certain type construction to prove that it is undecidable for finitely presented semigroups, with exception of only one property among those we consider. (The exception is the property of that every element in a semigroup is decomposable.) After that we ask whether this property is decidable for one-relation semigroups and one-relation monoids. For some properties like 'having an identity' this clearly should be separated between semigroup and monoid cases. The summary of results we put into Table 1.

We aim in this paper not only to prove some new results on decidability problems for finitely presented and one-relation semigroups, but also to survey already existing results in the area.

2 PRELIMINARIES

Let A be a finite alphabet and $\mathcal{R} \subseteq A^+ \times A^+$ a relation on the free semigroup A^+. By $\mathcal{R}^\#$ we denote the congruence on A^+ generated by $\mathcal{R}$, so that $\mathrm{Sg}\langle A \mid \mathcal{R}\rangle$ presents the semigroup [isomorphic to] $A^+/\mathcal{R}^\#$. Similarly, $\mathrm{Mon}\langle A \mid \mathcal{R}\rangle$ presents the monoid isomorphic to $A^*/\mathcal{R}^\#$, where $\mathcal{R}$ is now viewed as a relation on the free monoid A^*. The identity of the free monoid A^*, the empty word, we will denote throughout by ε. To avoid ambiguity, we also make a distinction between a word $u \in A^*$ and the element $\overline{u}$ it represents in the correspondent semigroup or monoid given by a presentation. So that $u = v$ will mean that u and v are equal as words; and $\overline{u} = \overline{v}$ will mean that u and v represent the same element of the semigroup (and may or may not be distinct as words). By *trivial relations* in a presentation we will mean pairs (w, w) for $w \in A^*$. We assume that the reader is familiar with the basic knowledge of semigroup presentations, for further background we refer the reader to [Ruš95] and [Hig92].

The set of natural numbers $\mathbb{N}$ is assumed *not* to contain 0; the semigroup free product is denoted by $*_S$ and the monoid free product by $*_M$. We will use later the fact that if $S = \mathrm{Sg}\langle A \mid \mathcal{R}\rangle$ and $T = \mathrm{Sg}\langle B \mid \mathcal{Q}\rangle$, then $S *_S T = \mathrm{Sg}\langle A, B \mid \mathcal{R}, \mathcal{Q}\rangle$, and that the same holds for monoid free products.

3 MARKOV PROPERTIES

In studying whether a given property is decidable for finitely presented semigroups, it turns out to be useful to know if it is a so-called Markov property, as such properties are undecidable for finitely presented semigroups. (By a 'property of semigroups' we mean those properties that are preserved under isomorphisms.) In most cases it is easy to check whether a particular property is a Markov property:

Table 1: Shows, for particular properties of semigroups, whether it or its negation is a Markov property and whether it is decidable for general finitely presented semigroups and one-relation semigroups. [In each column: Y = Yes, N = No, ? = Unknown; in the decidability columns: T = Always true, F = Always false].

PROPERTY $\mathfrak{P}$	MARKOV PROPERTY		DECIDABLE		
	$\mathfrak{P}$	$\neg\mathfrak{P}$	FIN. PRES.	ONE-RELATION	
				SEMIGROUP	MONOID
Finiteness	Y (Re. 4.1)	N (Th. 3.3)	N (Th. 3.2)	Y (Pr. 4.2)	Y (Pr. 4.2)
$S^2 = S$	N (Re. 5.2)	N (Re. 5.2)	Y (Pr. 5.1)	Y (Pr. 5.1)	T (Re. 5.2)
Having an identity	N (Pr. 6.1)	N (Pr. 6.1)	N (Pr. 6.3)	Y (Pr. 6.5)	T (By def.)
Having a zero	N (Pr. 6.2)	N (Pr. 6.2)	N (Pr. 6.4)	Y (Co. 6.7)	Y (Pr. 6.6)
Having idempotents	N (Th. 3.3)	Y (Trivial)	N (Th. 3.2)	Y (Th. 7.2)	T (By. def.)
Being a group	Y (Pr. 8.1)	N (Th. 3.3)	N (Th. 3.2)	Y (Pr. 8.6)	Y (Pr. 8.5)
Group-embeddability	Y (Pr. 8.1)	N (Th. 3.3)	N (Th. 3.2)	Y (Pr. 8.7)	Y (Pr. 8.8)
Cancellativity	Y (Pr. 8.10)	N (Th. 3.3)	N (Th. 3.2)	Y (Re. 8.11)	Y (Pr. 8.12)
Non-trivial subgroup	N (Th. 3.3)	Y (Trivial)	N (Th. 3.2)	?	?
Being inverse	Y (Pr. 8.1)	N (Th. 3.3)	N (Th. 3.2)	Y (Pr. 8.9)	?
Orthodoxy	Y (Pr. 8.1)	N (Th. 3.3)	N (Th. 3.2)	?	?
Regularity	?	N (Pr. 8.2)	N (Pr. 8.3)	Y (Pr. 8.4)	?
$\min_R$	Y (Pr. 9.1)	N (Th. 3.3)	N (Th. 3.2)	Y (Co. 9.3)	Y (Pr. 9.2)
Right-stability	Y (Pr. 9.1)	N (Th. 3.3)	N (Th. 3.2)	?	?
$\mathcal{J} = \mathcal{D}$	?	N (Pr. 10.1)	N (Pr. 10.2)	?	?
Being the bicyclic monoid	Y (Re. 11.3)	N (Th. 3.3)	N (Th. 3.2)	F (Co. 11.5)	Y (Pr. 11.4)
Being a BR-ext.	N (Pr. 11.1)	N (Pr. 11.1)	N (Pr. 11.2)	F (Co. 11.5)	?
Simplicity	N (Pr. 12.1)	N (Pr. 12.1)	N (Pr. 12.2)	Y (Co. 12.6)	?
Bisimplicity	?	N (Pr. 12.1)	N (Co. 12.5)	Y (Co. 12.6)	?
Semisimplicity	N (Pr. 12.3)	N (Pr. 12.3)	N (Pr. 12.4)	?	?
Proper direct product	N (Pr. 13.1)	N (Pr. 13.1)	N (Pr. 13.2)	Y (Pr. 13.3)	?
Proper free product	N (Pr. 14.1)	N (Pr. 14.1)	N (Pr. 14.3)	Y (Pr. 14.4)	?
Hopficity	N (Pr. 15.1)	N (Pr. 15.1)	N (Pr. 15.2)	?	?
Residual finiteness	Y (Pr. 16.1)	N (Th. 3.3)	N (Th. 3.2)	?	?

Definition 3.1. Let $\mathfrak{P}$ be a property of semigroups. Then $\mathfrak{P}$ is a *Markov property* if it satisfies the following three conditions:

1. There exists a finitely presented semigroup S_1 with property $\mathfrak{P}$.
2. There exists a finitely presented semigroup S_2 that does not embed into any finitely presented semigroup with property $\mathfrak{P}$.

The following theorem is folklore: the Markov properties are undecidable for finitely presented semigroups. The way *how* it is proved will help us later to prove undecidability results for finitely presented semigroups, so we include a sketch proof for the monoid case.

Theorem 3.2 ([BO93, Theorem 7.3.7]). *Let $\mathfrak{P}$ be a property of semigroups, and suppose that either $\mathfrak{P}$ or $\neg\mathfrak{P}$ is a Markov property. Then $\mathfrak{P}$ is undecidable for finitely presented semigroups.*

Proof of 3.2. Let $\mathfrak{P}$ be a Markov property and let S_1 and S_2 be two monoids as in Definition 3.1. Take any finitely presented monoid T with undecidable word problem. Set $S = T *_M S_2$. The monoid S has undecidable word problem and does not have property $\mathfrak{P}$. Also S is a finitely presented monoid: let $S = \mathrm{Mon}\langle A \mid \mathcal{R}\rangle$ be any finite presentation for S. Now for arbitrary $u, v \in A^*$ define

$$S_{u,v} = \mathrm{Mon}\langle A, c, d \mid \mathcal{R}, (cud, \varepsilon), (acvd, cvd) \quad (\forall a \in A \cup \{c, d\})\rangle\,. \tag{3.1}$$

The monoid $S_{u,v}$ satisfies the following two conditions: if $\overline{u} = \overline{v}$, then $S_{u,v}$ is trivial; if $\overline{u} \neq \overline{v}$ then S_2 embeds into $S_{u,v}$ and so $S_{u,v}$ does not have property $\mathfrak{P}$.

Thus whether the monoid $S_1 *_M S_{u,v}$ has property $\mathfrak{P}$ is equivalent to whether $\overline{u} = \overline{v}$. All the constructions used are effective, so the word problem reduces to the problem of deciding $\mathfrak{P}$: thus $\mathfrak{P}$ is undecidable.

If $\neg\mathfrak{P}$ is a Markov property, the result follows from the same proof on noting that $\mathfrak{P}$ is decidable if and only if $\neg\mathfrak{P}$ is decidable. $\boxed{3.2}$

Notice that the construction (3.1) appears in some other branches of mathematics, e.g. using it, Bernhard Neumann proved that every existentially closed monoid has only two congruences [LS77, Chapter IV]. Constructions of this type will be a principal tool for proving undecidability results for finitely presented semigroups later in the paper.

The following theorem, although not difficult, does not seem to have been explicitly stated hitherto:

Theorem 3.3. *Let $\mathfrak{P}$ be a property of semigroups. Then at most one of $\mathfrak{P}$ and $\neg\mathfrak{P}$ is a Markov property.*

Proof of 3.3. Suppose $\mathfrak{P}$ and $\neg\mathfrak{P}$ are both Markov properties. Then there exists a finitely presented semigroup S that does not embed into any finitely presented $\mathfrak{P}$-semigroup and a finitely presented semigroup T that does not embed into any finitely presented semigroup that is $\neg\mathfrak{P}$. Both S and T embed into $S *_S T$, which is finitely presented; thus $S *_S T$ can be neither $\mathfrak{P}$ nor $\neg\mathfrak{P}$. This is a contradiction, so at least one of $\mathfrak{P}$, $\neg\mathfrak{P}$ fails to be a Markov property. $\boxed{3.3}$

4 FINITE?

REMARK 4.1. Finiteness is obviously a Markov property, for no infinite semigroup can embed into a finite one.

PROPOSITION 4.2. *It is decidable whether a one-relation semigroup or monoid is finite.*

Proof of 4.2. Let $S = \mathrm{Mon}\langle A \mid (u, v)\rangle$. Suppose that A contains two distinct symbols a and b. If either u or v contains b, then by the Freiheitssatz for one-relation monoids (see [SW83]), a generates a free subsemigroup and so S is infinite. If neither u nor v contains b, then b generates a free subsemigroup.

Thus S is finite if and only if A is a singleton $\{a\}$ and the defining relation (u, v) is of the type (a^k, a^n) where $k, n \geqslant 0$ are distinct. $\boxed{4.2}$

5 $S^2 = S$?

In this section we consider the property of all elements of a semigroup being decomposable, abbreviated as '$S^2 = S$'. Given a generating set X for the semigroup S, the property $S^2 = S$ is equivalent to all elements of X being decomposable in S.

PROPOSITION 5.1. *There is an algorithm that takes as input a finite presentation for a semigroup S and decides whether $S^2 = S$.*

Proof of 5.1. Let $S = \mathrm{Sg}\langle A \mid \mathcal{R}\rangle$. If $\mathcal{R}$ contains any relations of the form (a, b) where $a, b \in A$ with $a \neq b$, use Tietze transformations to remove one of the two generators. So assume without loss of generality that $\mathcal{R}$ contains no such relations, and also no trivial relations.

It suffices to check whether every element of $\overline{A}$ is decomposable in S: that is, whether there exist $u, v \in A^+$ with $\overline{uv} = \overline{a}$ for each $a \in A$. For this to occur, there must be a sequence of $\mathcal{R}$-transitions from uv to a. Since $\mathcal{R}$ contains no relations of the form (a, b) for $a, b \in A$, this requires the presence in $\mathcal{R}$ of a relation (w, a) for each $a \in A$, where $|w| \geqslant 2$. Such relations are present if and only if every generator in $\overline{A}$ — and thus every element of S — is decomposable. $\boxed{5.1}$

REMARK 5.2. Notice that, as it is one of the few properties that are decidable for all finitely presented semigroups, $S^2 = S$ is not a Markov property. Note also that $S^2 = S$ obviously holds for all monoids.

Refining the reasonings from Proposition 5.1 to one-relation semigroups we obtain:

COROLLARY 5.3. *A one-relation semigroup $S = \mathrm{Sg}\langle A \mid (u, v)\rangle$ satisfies $S^2 = S$ if and only if $A = \{a\}$ and $u = a^k$ and $v = a$ (or vice versa), $k > 1$, in which case S is a finite cyclic group.*

6 IDENTITY? ZERO?

PROPOSITION 6.1. *Neither the property of having an identity, nor its negation, is a Markov property.*

PROPOSITION 6.2. *Neither the property of having a zero, nor its negation, is a Markov property.*

These results can be proved in parallel:

Proof of 6.2. Let S be a finitely presented semigroup. Then S embeds into the finitely presented semigroup S^1 (respectively, S^0); thus having an identity (respectively, a zero) is not a Markov property.

On the other hand, S embeds into the free product $S *_S \mathbb{N}$, which does not contain an identity (respectively, a zero); thus lacking an identity (respectively, a zero) is not a Markov property. $\boxed{6.2}$

Proposition 6.3. *It is undecidable whether a finitely presented semigroup has an identity.*

Proof of 6.3. Let $S = \mathrm{Sg}\langle A \mid \mathcal{R}\rangle$ be a semigroup with insoluble word problem. Take any $u, v \in A^*$ and construct a new semigroup

$$\begin{aligned} S'_{u,v} = \mathrm{Sg}\langle A, c, d, e \quad | \quad & \mathcal{R}, (cud, e), (acvd, cvd) \quad (\forall a \in A \cup \{c, d\}) \\ & (ae, a) \quad (\forall a \in A \cup \{c, d, e\})\rangle. \end{aligned}$$

If $\overline{u} = \overline{v}$ then every generator for $S'_{u,v}$ equals e, which is an idempotent, and so $S'_{u,v}$ is trivial.

If $\overline{u} \neq \overline{v}$ then $S'_{u,v}$ does not have an identity. To see this, suppose the contrary and observe that since e is a right identity, e would be this identity and thus we would have $ec = c$. Then, having that $e = cud \in \langle A, c, d\rangle$ and so $e = ecvd$, we would have $cud = e = ecvd = cvd$. This leads to a contradiction since $cud \neq cvd$ in the semigroup $S_{u,v}$ which is a homomorphic image of $S'_{u,v}$.

Therefore $S'_{u,v}$ contains an identity if and only if $\overline{u} = \overline{v}$. $\boxed{6.3}$

Proposition 6.4. *It is undecidable whether a finitely presented semigroup has a zero.*

Proof of 6.4. Notice that a group has a zero if and only if it is trivial. In addition, every finitely presented group is also a finitely presented semigroup. Hence, since 'being trivial' is a Markov property for groups, the claim follows immediately. $\boxed{6.4}$

Proposition 6.5. *It is decidable whether a one-relation semigroup has an identity.*

Proof of 6.5. Let $S = \mathrm{Sg}\langle A \mid (u, v)\rangle$ be a semigroup with identity $\overline{w}$. Then for every $a \in A$, we have that $\overline{a} = \overline{aw}$. Therefore there is a sequence of transitions from a to aw and so a is either u or v. So A contains at most two symbols. If $A = \{a, b\}$, then $u = a$ and $v = b$, whence $S \simeq \mathbb{N}$, which is a contradiction. If A is a singleton $\{a\}$ then, using elementary reasonings about monogenic semigroups, S has an identity if and only if the defining relation (u, v) is either (a, a^k) or (a^k, a) for some $k \geqslant 2$. $\boxed{6.5}$

Proposition 6.6. *It is decidable whether a one-relation monoid has a zero.*

Proof of 6.6. Let $S = \mathrm{Mon}\langle A \mid (u, v)\rangle$ be a monoid with a zero $\overline{w}$, where $w \in A^*$.

Assume first that $v = \varepsilon$. Then for any $a \in A$ we have $\overline{wa} = \overline{w}$. Thus there is a sequence of elementary transitions from wa to w. Since such a sequence can lead from wa only to words with length $|wa| + k|u|$ for some $k \in \mathbb{Z}$, we obtain $|u| = 1$. Thus a Tietze transformation can be used to remove a redundant generator from A and so S is a free monoid and does not contain a zero, unless it is trivial.

Now assume that $|u|, |v| \geqslant 1$. Obviously $w \in A^+$. Note that in a left-cancellative semigroup every idempotent is a left identity. So S cannot be left-cancellative and so, by Adjan's Theorem, u and v start with the same letter. Now, for any $a \in A$, the equality $\overline{aw} = \overline{w}$ holds. So there is a sequence of elementary transitions from aw to w. Thus w begins with the letter a. Since $a \in A$ was arbitrary, we obtain that $|A| = 1$. Therefore S contains a zero if and only if $A = \{a\}$ and (u, v) is (a^k, a^{k+1}) or (a^{k+1}, a^k) for some $k \in \mathbb{N}$. $\boxed{6.6}$

Since S contains a zero if and only if S^1 does, the following result is a corollary of the preceding proposition.

Corollary 6.7. *It is decidable whether a one-relation semigroup has a zero.*

7 idempotents?

Lallement proved in [Lal74] that it is decidable whether a one-relation semigroup contains an idempotent. We provide an alternative, shorter, proof for this.

We will need an auxiliary lemma which was proved in [LS62, Lemma 2]. We give a new proof of this lemma which, unlike the original one from [LS62], avoids inductive reasonings:

Lemma 7.1. *Let $s, x, y \in A^*$ be such that $sx = ys$ with $|x| = |y| < |s|$. Then there exist $u, v \in A^*$ and $n \in \mathbb{N}$ with $s = (uv)^n u$, $x = vu$ and $y = uv$.*

Proof of 7.1. Take the maximal n such that $s = y^n u$ for some $u \in A^*$. We have $y^n ux = y^{n+1} u$ and so $ux = yu$. So either y is a prefix of u or u is a prefix of y. By the choice of n, the former case is impossible. So $y = uv$ for some $v \in A^*$ and thus $s = (uv)^n u$. Therefore $(uv)^n ux = (uv)^{n+1} u$ and so $x = vu$. $\boxed{7.1}$

Theorem 7.2. *Let S be the one-relation semigroup $\mathrm{Sg}\langle A \mid (p, q)\rangle$ with $|p| \geqslant |q|$. Then S contains an idempotent if and only if one of the following two conditions hold:*

1. *$A = \{a\}$ (for some symbol a), $p = a^k$ (for some $k \geqslant 2$), and $q = a$, in which case S is a finite cyclic group.*
2. *S is neither left- nor right-cancellative, and q is both a prefix and a suffix of p.*

Proof of 7.2. First we note that if S contains an idempotent then $|p| \neq |q|$. Indeed, otherwise there would exist $w \in A^+$ such that the words ww and w represent the same element of S. But then, having $|p| = |q|$, we would have that $|ww| = |w|$, a contradiction.

Lemma 7.3. *If S is left-cancellative and contains an idempotent, then A contains a single symbol a, $p = a^k$ (for some $k \geqslant 2$), $q = a$, and thus S is a finite cyclic group.*

Proof of 7.3. Suppose S is left-cancellative and let $w \in A^+$ represent an idempotent of S. Then w is a left identity for S. So, for any $a \in A$, wa and a must represent the same element of S. Thus wa and a are linked by a sequence of elementary transitions. The last transition must have right-hand side a. That is, $q = a$ for every $a \in A$. Thus A must contain a single letter a, and $p = a^k$ for some $k \geqslant 2$ since $|p| > |q|$. $\boxed{7.3}$

Lemma 7.4. *If S is not left-cancellative (respectively, right-cancellative) and contains an idempotent and $|A| \geqslant 2$, then q is a prefix (respectively, suffix) of p.*

Proof of 7.4. If S is not left-cancellative, then p and q must start with the same letter. Let u be the longest common prefix of p and q, with $p = up'$ and $q = uq'$. Assume $q' \neq \varepsilon$. Then the semigroup $T = \mathrm{Sg}\langle A \mid (p', q')\rangle$ is left-cancellative and contains an idempotent since S does. By the previous lemma, $A = \{a\}$, which is a contradiction. $\boxed{7.4}$

Lemma 7.5. *If S is neither left- nor right-cancellative and $|A| \geqslant 2$, S has an idempotent if and only if q is both a prefix and a suffix of p.*

Proof of 7.5. The forward implication holds by the previous lemma.

Suppose q is both a prefix and suffix of p. If $|q| \leqslant (1/2)|p|$, then $p = qwq$ for some $w \in A^*$. In this case, qw represents an idempotent of S.

If $|q| > (1/2)|p|$, then $p = xq = qy$, and therefore $p = (uv)^{n+1}u$ and $q = (uv)^n u$ by Lemma 7.1. So $(uv)^{n+2}$ and $(uv)^{n+1}$ represent the same element of S, and hence $(uv)^{n+1}$ represents an idempotent of S. $\boxed{7.5}$

The above lemmas together imply the theorem. $\boxed{7.2}$

Corollary 7.6. *Let S be the one-relation semigroup $\mathrm{Sg}\langle A \mid (p, q)\rangle$. Assume without loss that $|p| \geqslant |q|$. Then S contains an idempotent if and only if q is a proper prefix and a proper suffix of p. In particular, it is decidable whether a one-relation semigroup has an idempotent.*

Question 7.7. Is it decidable whether a one-relation semigroup or monoid has infinitely many idempotents?

The following example shows that this question is not vacuous for one-relation semigroups:

Example 7.8. The semigroup $S = \mathrm{Sg}\langle a, b, c \mid (abaca, a)\rangle$ has infinitely many idempotents.

Proof of 7.8. For any $k \geqslant 1$ we have:

$$\overline{a} = \overline{abaca} = \overline{ababacaca} = \ldots = \overline{(ab)^k a (ca)^k}.$$

For each $k \in \mathbb{N}$, let $u_k = ba(ca)^k(ba)^{k-1}$. Notice that $(\{a, b, c\}, (abaca, a))$ is a confluent noetherian rewriting system. Thus the words u_k, being in normal form, represent distinct elements of S. Furthermore, each of the elements $\overline{u_k}$ is an idempotent:

$$\begin{aligned}
\overline{u_k}^2 &= \overline{ba(ca)^k(ba)^{k-1}}\,\overline{ba(ca)^k(ba)^{k-1}} \\
&= \overline{ba(ca)^{k-1}c}\,\overline{(ab)^k a(ca)^k}\,\overline{(ba)^{k-1}} \\
&= \overline{ba(ca)^{k-1}c}\,\overline{a}\,\overline{(ba)^{k-1}} \\
&= \overline{ba(ca)^k(ba)^{k-1}} \\
&= \overline{u_k}.
\end{aligned}$$

$\boxed{7.8}$

We start this section with recalling some definitions. A semigroup S is said to be *regular* if for every element $a \in S$ there exists $b \in S$ such that $a = aba$. A semigroup is *inverse* if it is regular and all its idempotents pairwise commute. Lastly, a semigroup is *orthodox* if it is regular and all the idempotents form a subsemigroup. It is obvious that every inverse semigroup is orthodox.

PROPOSITION 8.1. *Orthodoxy, being inverse, group-embeddability, and being a group are all Markov properties.*

Proof of 8.1. Group-embeddability is a Markov property since there are examples of finitely presented semigroups embeddable into groups and the semigroup $\mathrm{Sg}\langle a, b \mid (a^2, a), (b^2, b)\rangle$ is not embeddable into a group.

For the remaining three properties, orthodoxy is the weakest property, so it suffices to exhibit a finitely presented semigroup that does not embed into any finitely presented orthodox semigroup. Again, $\mathrm{Sg}\langle a, b \mid (a^2, a), (b^2, b)\rangle$ is not embeddable into any orthodox semigroup: it is generated by idempotents but does not consist entirely of idempotents. $\boxed{8.1}$

PROPOSITION 8.2. *Non-regularity is not a Markov property.*

Proof of 8.2. Follows from that any finitely presented regular semigroup S embeds into the finitely presented non-regular semigroup $S *_S \mathbb{N}$. $\boxed{8.2}$

It remains an open problem whether regularity is a Markov property. Related results to this one can find in [BM02]. However we can prove that regularity is undecidable for general finitely presented semigroups:

PROPOSITION 8.3. *Regularity is undecidable for finitely presented semigroups.*

Proof of 8.3. Take an arbitrary finitely presented semigroup $S = \mathrm{Sg}\langle A \mid \mathcal{R}\rangle$ with insoluble word problem and indecomposable generators (e.g., Tseitin's semigroup, see [AD00, Theorem 2.2]). Pick $u, v \in A^+$. If $\overline{u} = \overline{v}$ then the semigroup $S_{u,v}$ (as defined by Eq. (3.1)) is trivial and so is regular. Suppose, with the aim of obtaining a contradiction, that $\overline{u} \neq \overline{v}$ and $S_{u,v}$ is regular. Then for any letter a, there exists a word $w \in (A \cup \{c, d\})^*$ such that $\overline{a} = \overline{awa}$.

We will prove now by induction on k that every chain of transitions $a = w_0 \to w_1 \to \cdots \to w_k$ is such that none of w_i contains a factor cpd where $\overline{p} = \overline{v}$ and $p \in (A \cup \{c, d\})^*$. The base case is obvious since $\overline{u} \neq \overline{v}$. Assume that for chains of lengths $\leqslant k$ the hypothesis holds and take a chain $a = w_0 \to w_1 \to \cdots \to w_k \to w_{k+1}$ contradicting the hypothesis. Then w_{k+1} contains a factor cpd with $\overline{p} = \overline{v}$. Clearly the transition $w_k \to w_{k+1}$ cannot correspond to a relation from $\mathcal{R} \cup \{(bcvd, cvd)\}$. If $w_k \to w_{k+1}$ corresponded to the insertion of the word cud, then w_k would contain the factor cpd. Hence $w_k \to w_{k+1}$ corresponds to the deletion of the word cud. Then w_k must contain a factor cp_1cudp_2d with $p_1p_2 = p$. This is a contradiction since $\overline{p_1cudp_2} = \overline{v}$. Thus the statement of induction is proved.

In any chain of transitions from a to awa there can be used only insertions or deletions of cud and relations from $\mathcal{R}$. Hence by a routine induction one shows that $w = cpd$ for some $p \in (A \cup \{c, d\})^*$. Let $a = w_0 \to w_1 \to \cdots \to w_{k-1} \to w_k = acpda$ be a chain from a to a word of the type $acpda$ with minimal possible length. If $w_{k-1} \to w_k$ is the insertion of cud, then $w_{k-1} \in ac(A \cup \{c, d\})^* da$, a contradiction. If $w_{k-1} \to w_k$ corresponds to a relation from $\mathcal{R}$, then, since a is indecomposable element in S, we again have $w_{k-1} \in ac(A \cup \{c, d\})^* da$. Therefore $w_{k-1} \to w_k$ is the deletion of cud. Hence either $w_{k-1} = cudawa$ or $w_{k-1} = awacud$, without loss of generality we may assume the first case. In order to reach $cudawa$ from a, there must exist r such that $w_{r+1} = cudw_r$. Find the largest such r. Then $w_i = cudw'_i$ for every i with $r+1 \leqslant i \leqslant k-1$. Then $a = w_0 \to w_1 \to \cdots w_r \to w'_{r+2} \to \cdots w'_{k-1} = awa$ is a chain with length shorter than the initial one, a contradiction.

Thus, $S_{u,v}$ is regular if and only if $\overline{u} = \overline{v}$ in S. $\boxed{8.3}$

Although we do not know the answer to the question on decidability of regularity for one-relation monoids, we provide it for one-relation semigroups:

Proposition 8.4. *It is decidable whether a one-relation semigroup is regular.*

Proof of 8.4. Take an arbitrary one-relation semigroup $S = \mathrm{Sg}\langle A \mid (u, v)\rangle$ and assume that it is regular. Since every generator must be decomposable, we have that either A is a singleton or that the relation is of the type (a, b) for some $a, b \in A$. The latter case is obviously impossible. So that S is regular if and only if A is a singleton $\{a\}$ and the relation is of the type (a^k, a) or (a, a^k) for some $k \geqslant 2$. $\boxed{8.4}$

Proposition 8.5. *It is decidable whether a one-relation monoid is a group.*

Proof of 8.5. Consider an arbitrary one-relation monoid $S = \mathrm{Mon}\langle A \mid (u, v)\rangle$.

Suppose first that both u and v are non-empty. Then no sequence of elementary transitions links the empty word ε and a non-empty word. Thus no non-empty word u can represent an invertible element, for if $v \in A^*$ represents its inverse, there would have to be a transition sequence from uv to ε. So in this case, S can never be a group.

Suppose now, without loss of generality, that $v = \varepsilon$. Then S is a group if and only if every letter from the alphabet is both right and left invertible. This is equivalent to that every letter from the alphabet divides ε both from left and right, and the division problem for one-relation special monoids is soluble [Adj66]. $\boxed{8.5}$

Proposition 8.6. *It is decidable whether a one-relation semigroup is a group.*

Proof of 8.6. If a semigroup is a group, it contains an identity. By the proof of Proposition 6.5, a one-relation semigroup contains an identity if and only if it is a group. This is possible only in the case when $A = \{a\}$ and the relation is (a^k, a) or (a, a^k) where $k \geqslant 2$. $\boxed{8.6}$

Proposition 8.7. *It is decidable whether a one-relation semigroup is group-embeddable.*

Proof of 8.7. If A is a singleton $\{a\}$ then a semigroup $\mathrm{Sg}\langle A \mid (u, v)\rangle$ is group-embeddable if and only if the relation is of the type (a, a^k) or (a^k, a) for some $k \geqslant 1$. If $|A| > 1$ then, by Adjan's theorem [Adj66], $\mathrm{Sg}\langle A \mid (u, v)\rangle$ embeds into a group if and only if u and v start and end with different letters. $\boxed{8.7}$

Proposition 8.8. *It is decidable whether a one-relation monoid is group-embeddable.*

Proof of 8.8. Let $S = \mathrm{Mon}\langle A \mid (u,v)\rangle$ be a one-relation monoid. Suppose that $u, v \in A^+$. Then $S \simeq T^1$, where $T = \mathrm{Sg}\langle A \mid (u,v)\rangle$, and S is group-embeddable if and only if T is group-embeddable and does not contain an identity: both of these properties are decidable by Propositions 8.7 and 6.5.

Now suppose that S is a special monoid with, say, $v = \varepsilon$. We will show that S is group embeddable if and only if $\mathrm{Mon}\langle \mathrm{cont}(u) \mid (u, \varepsilon)\rangle$ is a group where $\mathrm{cont}(u)$ is the content of the word u. To see this, let $u = pq$ for some $p, q \in A^*$. Then $\overline{pq} = \overline{\varepsilon}$ and so $\overline{qp} = \overline{\varepsilon}$, for otherwise S would contain a copy of the bicyclic monoid (see [CP61, Lemma 1.31]) and so could not be group-embeddable. Therefore every letter from $\mathrm{cont}(u)$ is invertible and so the claim follows. The sufficiency is obvious. It remains to use Proposition 8.5. $\boxed{8.8}$

Unfortunately we do not know if it is decidable whether a one-relation monoid is inverse, but we can do it for one-relation semigroups:

Proposition 8.9. *It is decidable whether a one-relation semigroup is inverse.*

Proof of 8.9. Take an arbitrary one-relation semigroup $S = \mathrm{Sg}\langle A \mid (u,v)\rangle$ and suppose that it is inverse.

Suppose first that $u, v \in A^+$. The semigroup S contains idempotents and so by Theorem 7.2 we have that u and v start with the same letter. Take now an arbitrary $a \in A$ and $w \in A^+$. Then we have

$$\overline{a}\,\overline{a}^{-1}\overline{w}\,\overline{w}^{-1} = \overline{w}\,\overline{w}^{-1}\overline{a}\,\overline{a}^{-1},$$

and so w must start with a. This means that A is a singleton and, using elementary reasonings about monogenic semigroups, S must be a group.

Thus a one-relation semigroup is an inverse semigroup if and only if it is a group. The result now follows from Proposition 8.6. $\boxed{8.9}$

The last property of this section we discuss is the cancellativity. First we deal with the general finitely presented semigroups case:

Proposition 8.10. *Cancellativity is a Markov property.*

Proof of 8.10. Obviously there are examples of cancellative finitely presented semigroups. On the other hand the bicyclic monoid is not embeddable into a cancellative finitely presented semigroup. $\boxed{8.10}$

Remark 8.11. It is a classical result of Adjan [Adj66] that a one-relation semigroup $\mathrm{Sg}\langle A \mid (u,v)\rangle$ is cancellative if and only if either A is a singleton, or u and v start with different letters and end with different letters.

Proposition 8.12. *It is decidable whether a one-relation monoid is cancellative.*

Proof of 8.12. Take a one-relation monoid $S = \mathrm{Mon}\langle A \mid (u,v)\rangle$. If $|u|, |v| \geqslant 1$ then S is cancellative if and only if $\mathrm{Sg}\langle A \mid (u,v)\rangle$ is, and so it remains to use Remark 8.11. If, say, $v = \varepsilon$ then S is cancellative if and only if $S' = \mathrm{Mon}\langle \mathrm{cont}(u) \mid (u, \varepsilon)\rangle$ is a group. The sufficiency is obvious. For the other direction, using the same methods as in the proof of Proposition 8.8, we prove that either S' is a group or S contains a copy of the bicyclic monoid. In the latter case S cannot be cancellative. It remains to use Proposition 8.5. $\boxed{8.12}$

9 Right stable? $\min_{\mathcal{R}}$?

To begin this section, we start with recalling some definitions. There is a natural order on the $\mathcal{R}$-classes of a semigroup S: we say that $R_a \leqslant R_b$ if and only if $a \in bS^1$. A semigroup is said to have the property $\min_{\mathcal{R}}$ if every infinite descending chain of $\mathcal{R}$-classes eventually stabilises.

A semigroup is said to be *right stable* if for every $\mathcal{D}$-class D of S the set X of all $\mathcal{R}$-classes in D has a minimal element (inside X) with respect to the order described in the previous paragraph (in such a case, every element in X is minimal inside X). An alternative characterisation of right stability is that for any $a, b \in S$, $R_a \leqslant R_{ba}$ implies $R_a = R_{ba}$, see [Lal79].

Obviously if a semigroup has $\min_{\mathcal{R}}$ then it is right stable.

Proposition 9.1. *Right stability and* $\min_{\mathcal{R}}$ *are Markov properties.*

Proof of 9.1. Since right stability is the weaker property among the two, it suffices to exhibit an example of a finitely presented semigroup, which cannot embed into a finitely presented right stable semigroup.

Take the bicyclic monoid $B = \mathrm{Mon}\langle b, c \mid (cb, \varepsilon)\rangle$. Assume that B is embeddable into a finitely presented right stable semigroup S. We have a strictly descending chain of $\mathcal{R}$-classes in B:

$$R^B_b > R^B_{b^2} > R^B_{b^3} > \cdots .$$

This gives a descending chain of $\mathcal{R}^S$-classes in S: $R^S_b \geqslant R^S_{b^2} \geqslant R^S_{b^3} \geqslant \cdots$. This latter chain must stabilise since all the $\mathcal{R}$-classes from it come from the same $\mathcal{J}^B$-class and so from the same $\mathcal{J}^S$-class. So that there exist $k < n$ such that $R^S_{b^k} = R^S_{b^n}$. Now, since B is a regular subsemigroup of S, we have by [How95, Proposition 2.4.2] that $\mathcal{R}^B = \mathcal{R}^S \cap (B \times B)$. This means that $R^B_{b^k} = R^B_{b^n}$, a contradiction.

Thus right stability is a Markov property. $\boxed{9.1}$

We do not know whether it is decidable for one-relation semigroups or monoids to be right stable. However we prove that the correspondent question for the property $\min_{\mathcal{R}}$ is decidable:

Proposition 9.2. *It is decidable whether a one-relation monoid has* $\min_{\mathcal{R}}$.

Proof of 9.2. Take an arbitrary one-relation monoid $S = \mathrm{Mon}\langle A \mid (u, v)\rangle$.

Assume first that both u and v come from A^+. The aim is to show that S has $\min_{\mathcal{R}}$ if and only if A is a singleton $\{a\}$ and the relation has the form (a^k, a^n) for distinct k and n. The sufficiency is obvious. So assume that S has $\min_{\mathcal{R}}$ and suppose, with the aim of obtaining a contradiction, that A contains distinct letters a and b. Let $d \in \{a, b\}$. Then, since the chain $R_d \geqslant R_{d^2} \geqslant R_{d^3} \geqslant \cdots$ must stabilise, we have that there exist $k < n$ and $x \in A^*$ such that $\overline{d^k} = \overline{d^n x}$. This implies that one of u and v is a power of d. Therefore, interchanging u and v if necessary, $u = a^p$ and $v = b^q$. If either p or q is equal to 1, then S is isomorphic to $\mathbb{N} \cup \{0\}$. If, on the other hand, $p, q \geqslant 2$ then we obtain a strict descending chain $R_{ab} > R_{(ab)^2} > R_{(ab)^3} > \cdots$. In either case, we have a contradiction. Hence, $A = \{a\}$ and so the relation is as above (for otherwise S would be isomorphic to $\mathbb{N} \cup \{0\}$ and so not $\min_{\mathcal{R}}$).

Assume now that $v = \varepsilon$. The aim is to show that S is $\min_{\mathcal{R}}$ if and only if it is a group, the sufficiency being obvious. Then this would imply, in view of Proposition 8.5, that $\min_{\mathcal{R}}$ is decidable for one-relation monoids. So, suppose that S has $\min_{\mathcal{R}}$. Let $u = pa$ where $a \in A$. Consider the chain $R_a \geqslant R_{a^2} \geqslant R_{a^3} \geqslant \cdots$. Since it must stabilise, there exist $k < n$ and $x \in S$ such that $\overline{a^k} = \overline{a^n x}$. This implies $\overline{\varepsilon} = \overline{p^k a^k} = \overline{p^k a^n x} = \overline{a^{n-k} x}$ and so a is invertible. Hence $\overline{p}$ is invertible and similarly the last letter from p represents an invertible element. Continuing in this way, one sees that all the letters from u represent invertible elements. Obviously all the letters from A must appear in u. Thus every generator is invertible and so S is a group, as required. $\boxed{9.2}$

Corollary 9.3. *It is decidable whether a one-relation semigroup has* $\min_{\mathcal{R}}$.

Proof of 9.3. It is easy to see that a semigroup S has $\min_{\mathcal{R}}$ if and only if S^1 has. So that $\mathrm{Sg}\langle A \mid (u,v)\rangle$ has $\min_{\mathcal{R}}$ if and only if $\mathrm{Mon}\langle A \mid (u,v)\rangle$ has. The statement now follows from Proposition 9.2. $\boxed{9.3}$

10 $\mathcal{J} = \mathcal{D}$?

We do not know whether the property of having $\mathcal{J} = \mathcal{D}$ is a Markov property. However we prove that its negation is not:

Proposition 10.1. *The negation of $\mathcal{J} = \mathcal{D}$ is not a Markov property.*

Proof of 10.1. Take an arbitrary finitely presented semigroup S. If T is an arbitrary finitely presented monoid with $\mathcal{J}^T \neq \mathcal{D}^T$ (for example $T = \mathrm{Mon}\langle a,b,c \mid (abc,\varepsilon)\rangle$, see [Lal79, Excercise 9, Chapter 2]) then $S^1 \times T$ is finitely presented, contains S and does not possess $\mathcal{J} = \mathcal{D}$. $\boxed{10.1}$

Proposition 10.2. *The property of having $\mathcal{J} = \mathcal{D}$ is undecidable for finitely presented semigroups.*

Proof of 10.2. As in the proof of Proposition 8.3, consider an arbitrary finitely presented semigroup $S = \mathrm{Sg}\langle A \mid \mathcal{R}\rangle$ with insoluble word problem and indecomposable generators. Pick $u, v \in A^+$. If $\overline{u} = \overline{v}$ then $S_{u,v}$ is trivial and so has $\mathcal{J} = \mathcal{D}$. So suppose $\overline{u} \neq \overline{v}$. Any any letter a, appearing in u, is $\mathcal{J}$-related to ε. We claim that $(\overline{a}, \varepsilon) \notin \mathcal{D}$. Indeed, if $\overline{a}\mathcal{D}\varepsilon$ then a is regular in $S_{u,v}$ and so we would be able to find $w \in S_{u,v}$ such that $\overline{a} = \overline{awa}$. As in the proof of Proposition 8.3, this is a contradiction. $\boxed{10.2}$

We do not know whether $\mathcal{J} = \mathcal{D}$ is decidable for one-relation semigroups or monoids. Possible hints could be taken from [Lal74], where one-sided and two-sided divisibility problems are solved for some important classes of one-relation monoids.

11 A Bruck–Reilly extension? Bicyclic?

Recall that the *Bruck–Reilly extension* of a monoid $M = \mathrm{Mon}\langle A \mid \mathcal{R}\rangle$ with respect to a (monoid) endomorphism $\vartheta : M \to M$ is the monoid

$$\mathrm{BR}(M,\vartheta) = \mathrm{Mon}\langle A, b, c \mid \mathcal{R}, (bc,\varepsilon), (ac, c(a\vartheta)), (ba, (a\vartheta)b) \quad (\forall a \in A)\rangle\,,$$

where b, c are new symbols not in A and $a\vartheta$ is interpreted as some fixed word in A^* representing $\overline{a}\vartheta$. If S is a semigroup without an identity and ϑ is an endomorphism of S, then the Bruck–Reilly extension of S with respect to ϑ is defined to be $\mathrm{BR}(S^1, \vartheta^*)$, where $\vartheta^* : S^1 \to S^1$ is defined by $s \mapsto s\vartheta$ for all $s \in S$ and $1 \mapsto 1$. The bicyclic monoid B is the Bruck–Reilly extension of the trivial monoid:

$$B = \mathrm{Mon}\langle b, c \mid (bc, \varepsilon)\rangle .$$

Note that any semigroup S embeds into any of its Bruck–Reilly extensions. If W is a set of canonical forms for M, then the set $\{c^k w b^n : k, n \geqslant 0,\ w \in W\}$ forms the canonical forms for $\mathrm{BR}(M, \vartheta)$. It turns out that $\mathrm{BR}(M, \vartheta)$ is isomorphic to the semigroup of triples (k, m, n) (where $k, n \geqslant 0$ and $m \in M$) subject to the multiplication

$$(k, m_1, n) \cdot (p, m_2, q) = (k - n + r, (m_1\vartheta^{r-n})(m_2\vartheta^{r-p}), q - p + r),$$

where $r = \max(n, p)$.

Let $\mathfrak{P}$ be the property of being a Bruck–Reilly extension.

Proposition 11.1. *Neither $\mathfrak{P}$ nor $\neg\mathfrak{P}$ is a Markov property.*

Proof of 11.1. Let S be a finitely presented semigroup. Then S embeds into $S *_S \mathbb{N}$, which is finitely presented and $\neg\mathfrak{P}$, for the latter semigroup has no identity. On the other hand, S embeds into $\mathrm{BR}(S^1, \vartheta^*)$ which is manifestly $\mathfrak{P}$. $\boxed{11.1}$

Nonetheless, $\mathfrak{P}$ is undecidable for the general finitely presented semigroups.

Proposition 11.2. *For finitely presented semigroups, $\mathfrak{P}$ is undecidable.*

Proof of 11.2. Let $S = \mathrm{Sg}\langle A \mid \mathcal{R}\rangle$ be a finitely presented semigroup with unsolvable word problem. Pick $u, v \in A^+$ and construct

$$\begin{aligned} S_{u,v,x,y} &= \mathrm{Mon}\langle A, c, d, x, y \mid \mathcal{R}, (xy, \varepsilon), (cud, \varepsilon), (bcvd, cvd) \text{ where } b \in A \cup \{c, d\}\rangle \\ &= S_{u,v} *_M \mathrm{Mon}\langle x, y \mid (xy, \varepsilon)\rangle . \end{aligned}$$

We will now prove that $S_{u,v,x,y}$ is a Bruck–Reilly extension precisely when $S_{u,v}$ is trivial (in which case $S_{u,v,x,y}$ is the bicyclic monoid). The sufficiency is obvious.

Suppose now that $S_{u,v}$ is not trivial and $S_{u,v,x,y}$ is a Bruck–Reilly extension $\mathrm{BR}(M, \vartheta)$ so that every element is a triple (t^k, m, r^n) such that $rt = \varepsilon$ and $\langle r, t\rangle$ is the bicyclic monoid, and $m \in M$. Since $xy = \varepsilon$, we obtain that $x = (1, m_1, r^k)$ and $y = (t^k, m_2, 1)$ for some $k \geqslant 0$ and $m_1, m_2 \in M$ with $m_1 m_2 = 1_M$. We have two cases to consider:

Case 1. $k > 0$. Let w represent a canonical form in the free product for $(t, 1_M, r)$. Then since $(t, 1_M, r) \cdot (t^k, m_2, 1) = (t^k, m_2, 1)$, we have that $wy = y$. Thus $w \in \mathrm{Mon}\langle x, y \mid (xy, \varepsilon)\rangle$ and either $w = \varepsilon$ or $w = yx$. The first case is impossible, for w does not represent ε. Hence $yx = w = (t, 1_M, r)$. On the other hand, $yx = (t^k, m_2 m_1, r^k)$. So that $k = 1$ and $m_2 m_1 = 1_M$. Note that $\overline{1_M} = \varepsilon$.

We also have that there are no invertible elements in $S_{u,v}$ except ε. Indeed, in every chain of transitions from ε to a word p there can be used only the relations from $\mathcal{R} \cup \{(cud, \varepsilon)\}$. Hence, if $p \neq \varepsilon$, then p starts with c, ends with d and the corresponding subsequence of c's and d's in p forms the correct bracketing sequence. Thus if $\overline{w_1 w_2} = \overline{w_2 w_1} = \overline{\varepsilon}$ and $\overline{w_1} \neq \overline{\varepsilon}$, then $|w_1|_c > |w_2|_c$ and $|w_2|_c > |w_2|_c$, a contradiction.

Now, in both components $S_{u,v}$ and $\mathrm{Mon}\langle x, y \mid (xy, \varepsilon)\rangle$ there are no invertible elements except ε, hence $S_{u,v,x,y}$ does not have invertible elements but ε. Hence $m_1 = m_2 = 1$. So, $x = (1, 1, r)$ and $y = (t, 1, 1)$. Thus $yx = (t, 1, r)$. Since $S_{u,v}$ is not trivial, we have that M is not trivial (otherwise $S_{u,v,x,y}$ would coincide with $\mathrm{Mon}\langle x, y \mid (xy, \varepsilon)\rangle$). Take $m \in M \setminus \{1\}$. Now,

$$(t, m\vartheta, r) = (t, 1, r) \cdot (1, m, 1) = (1, m, 1) \cdot (t, 1, r).$$

So, if $w_0 = c_1 \cdots c_p$ is the normal form for $(1, m, 1)$ then $c_1 \in \langle x, y\rangle$. Moreover, we have that $yxc_1 = c_1$. Then $c_1 = y^e x^f$ for some $e \geqslant 1$ and $f \geqslant 0$. This implies that w_0 represents (t^g, m', r^h) for some $g \geqslant e$, a contradiction.

Case 2. $k = 0$. Then we have $x = (1, m_1, 1)$ and $y = (1, m_2, 1)$ with $m_1 m_2 = 1_M$. Notice that

$$(1, m_2 m_1, 1) \cdot (t, (m_2 m_1)\vartheta, r) = (t, (m_2 m_1)\vartheta \cdot (m_2 m_1)\vartheta, r) = (t, (m_2 m_1)\vartheta, r).$$

Let $w = c_1 \cdots c_p$ be the normal form for $(t, (m_2 m_1)\vartheta, r)$. Since $(1, m_2 m_1, 1) = yx$, we have that $c_1, c_p \in \langle x, y\rangle$. Now notice that $(t, (m_2 m_1)\vartheta, r)$ is an idempotent. Hence $c_p c_1 = \varepsilon$ and so $c_1 = y^n$ and $c_p = x^n$ for some $n \geqslant 1$. So $w = y^n w_0 x^n$ for some w_0 which starts and ends with a component from $S_{u,v}$. Now,

$$w_0 = x^n w y^n = (t, (m_1^n)\vartheta(m_2 m_1)\vartheta(m_2^n)\vartheta, r) = (t, (1_M)\vartheta, r) = (t, 1_M, r).$$

Therefore $yx \cdot w_0 = w_0 \cdot yx$, a contradiction.

Thus the word problem for finitely presented semigroups reduces to the question of being $\mathfrak{P}$ for finitely presented semigroups; thus the latter is undecidable. $\boxed{11.2}$

Remark 11.3. 'Being the bicyclic monoid' *is* a Markov property: for example, no non-trivial finitely presented group embeds into the bicyclic monoid.

Proposition 11.4. *The bicyclic monoid admits a unique one-relation monoid presentation (up to relabelling of generators and exchanging the two sides of the defining relation), namely* $\mathrm{Mon}\langle b, c \mid (bc, \varepsilon)\rangle$.

Before embarking on the proof, witness that this result implies that it is decidable whether a one-relation monoid is the bicylic monoid.

Proof of 11.4. Let $\mathfrak{S} = \mathrm{Mon}\langle A \mid (u, v)\rangle$ be a presentation for the bicyclic monoid B.

If both u and v come from A^+, then the identity of B would not be possible to decompose in a non-trivial way, a contradiction.

So assume that $v = \varepsilon$. The alphabet A must contain at least two symbols. Furthermore, all the letters from A appear in u, since otherwise $\mathfrak{S}$ would present a proper free product, which is a contradiction. If $|A| > 2$, by the Freiheitssatz for one-relation monoids [SW83], we have that any two elements of $\overline{A}$ generate a free submonoid of B. But from Descalço and Ruškuc's description of all subsemigroups of B [DR05], it follows that B does not contain a 2-generated free subsemigroup. Therefore A is a 2-set.

Suppose that $A = \{b, c\}$. Assume without loss of generality that $u = pc$. Clearly, $p \neq \varepsilon$. If p starts with c then c is right- and left-invertible and, since the only invertible element in B is the identity, B is monogenic, which is a contradiction. So $p = bqc$ for some $q \in A^*$ and the relation has the form (bqc, ε). Since the monoid presentation $\mathrm{Mon}\langle b, c \mid (bqc, \varepsilon)\rangle$ presents B, the group presentation $\mathrm{Gp}\langle b, c \mid (bqc, \varepsilon)\rangle$ presents $\mathbb{Z}$ [CP61, Corollary 1.32].

Suppose that u is not of the form $b^k c^n$ where $k, n \geqslant 0$. Then $u = b^{k_1}c^{n_1} \cdots b^{k_s}c^{n_s}$ for some $k_i, n_i \geqslant 1$. By [LS77, Lemma V.11.8], $\mathbb{Z} = \mathrm{Gp}\langle b, c \mid (bqc, \varepsilon)\rangle$ has a presentation $\mathfrak{H} = \mathrm{Gp}\big\langle x, y \mid (y^{l_1}x^{m_1} \cdots y^{l_t}x^{m_t}, \varepsilon)\big\rangle$ for some $t \geqslant s$, $l_i, m_i \in \mathbb{Z} \setminus \{0\}$, and such that either x or y has zero exponent sum. But in this case, $\mathfrak{H}$ will present an HNN-extension of a non-trivial group (see [LS77, Chapter IV.5]), and so cannot be $\mathbb{Z}$, which is a contradiction.

Thus $u = b^k c^n$ for some $k, n \in \mathbb{N}$. The aim is now to complete the proof by proving that $k = n = 1$.

Suppose that $n > 1$. Then $\overline{c^n b^k}$ and $\overline{cb^k c^{n-1}}$ are idempotents and, so since B is an inverse monoid, must commute. Therefore

$$\begin{aligned}
&\overline{c^n b^k} \\
&= \overline{cb^k c^n c^{n-1} b^k} && \text{(by the defining relation } (b^k c^n, \varepsilon)) \\
&= \overline{cb^k c^{n-1} c^n b^k} \\
&= \overline{c^n b^k c b^k c^{n-1}} && \text{(by the commutativity of the idempotents).}
\end{aligned}$$

But $\big(\{b, c\}, (b^k c^n, \varepsilon)\big)$ is a confluent rewriting system and $c^n b^k$ and $c^n b^k c b^k c^{n-1}$ are in normal forms but not equal: this is a contradiction. Therefore $n = 1$. Analogously, one can prove that $k = 1$. $\boxed{11.4}$

The following result follows from the proof of Proposition 6.5:

Corollary 11.5. *A one-relation semigroup is never a Bruck–Reilly extension.*

We conjecture that a one-relation monoid is a Bruck–Reilly extension if and only if it is the bicyclic monoid.

12 SIMPLE? BISIMPLE? SEMISIMPLE?

Recall that a semigroup S is *simple* if it has no ideals other than S itself; it is *bisimple* if it consists of a single $\mathcal{D}$-class. A semigroup S with a zero 0 is 0-*simple* if $S^2 \neq \{0\}$ and its only ideals are S and $\{0\}$. It is *semisimple* if every principal factor of S is 0-simple or simple. For further information about these notions we refer the reader to [CP61].

Proposition 12.1. *Neither simplicity nor non-simplicity is a Markov property; non-bisimplicity is not a Markov property.*

Proof of 12.1. Every semigroup S embeds into the non-simple semigroup S^0. Thus neither non-simplicity nor non-bisimplicity is a Markov property.

On the other hand, every semigroup S embeds into the Bruck–Reilly extension $\mathrm{BR}(S^1, \vartheta)$, where $\vartheta : S^1 \to S^1$ is the trivial endomorphism: $s\vartheta = 1$ for all $s \in S^1$. Since $S^1\vartheta$ is contained in the group of units of S^1, the extension $\mathrm{BR}(S^1, \vartheta)$ is simple [How95, Proposition 5.6.6(1)]. $\boxed{12.1}$

The question on whether bisimplicity is a Markov property remains open.

Proposition 12.2. *Simplicity is undecidable for finitely presented semigroups.*

Proof of 12.2. Let S be a finitely presented group with unsolvable word problem and let $S = \mathrm{Mon}\langle A \mid \mathcal{R}\rangle$ be a finite monoid presentation for S. Pick $v \in A^+$ and consider the monoid $S_{\varepsilon,v}$. Let $I_{\varepsilon,v} = S_{\varepsilon,v}\overline{cvd}S_{\varepsilon,v}$. If $\overline{\varepsilon} = \overline{v}$, then $S_{\varepsilon,v}$ is trivial and so simple.

So let $\overline{\varepsilon} \neq \overline{v}$. We will prove that $\overline{\varepsilon} \notin I_{\varepsilon,v}$, which will yeild that S is non-simple. So, assume that $\overline{\varepsilon} \in I_{\varepsilon,v}$. Then there exist p, q such that $\overline{pcvdq} = \overline{\varepsilon}$. Then $\overline{cvdq} = \overline{\varepsilon}$ and so $\overline{a} = \overline{acvdq} = \overline{cvdq} = \overline{\varepsilon}$ for all $a \in A \cup \{c, d\}$, a contradiction.

Thus the word problem for finitely presented groups reduces to the question of simplicity for finitely presented semigroups. $\boxed{12.2}$

Recall that if S is semisimple and I is an ideal in S then every ideal in I is an ideal in S, see [CP61, Theorem 2.41].

Proposition 12.3. *Neither semisimplicity, nor non-semisimplicity, is a Markov property.*

Proof of 12.3. Let S be a finitely presented semigroup. Then S embeds into the simple finitely presented semigroup $BR(S^1, \vartheta)$, where $s\vartheta = 1$ for all $s \in S$ [How95, Proposition 5.6.6(1)]. It remains to note that every simple semigroup is semisimple.

Let T be any non-semisimple finitely presented monoid (for example $T = \mathbb{N}_0$: it has an ideal $\{2, 3, \cdots\}$, in which $\{3, 5, 6, 7, \cdots\}$ is an ideal, and in T not). Then a finitely presented semigroup $S^1 \times T^1$ contains S and is not semisimple. $\boxed{12.3}$

Proposition 12.4. *Semisimplicity is undecidable for finitely presented semigroups.*

Proof of 12.4. Let $S = \mathrm{Sg}\langle A \mid \mathcal{R}\rangle$ be any finitely presented semigroup with insoluble word problem. Pick $u, v \in A^+$. If $\overline{u} = \overline{\text{Solet}v}$ then $S_{u,v}$ is trivial and so semisimple.

So let $\overline{u} \neq \overline{v}$. Consider the ideal $I = S_{u,v}\overline{cd}S_{u,v}$ in $S_{u,v}$. Take the ideal $J = I\overline{cd}I$ in I. We will prove that $\overline{c^2d} \notin J$ and this will complete the proof. Suppose that $\overline{c^2d} \in J$, i.e. $\overline{c^2d} \in \overline{X^*cdX^*cdX^*cdX^*}$ where $X = A \cup \{c, d\}$. By a straightforward induction on lengths of chains, it follows that any chain of transitions starting from c^2d cannot lead to a word with a factor cpd such that $\overline{p} = \overline{v}$. Take now any chain $c^2d = w_0 \to w_1 \to \cdots \to w_{k-1} \to w_k$ to a word w_k of the form $p_1cdp_2cdp_3cdp_4$ with shortest possible length. Without loss of generality we may assume that p_1, p_2 and p_3 do not contain the factor cd. Each transition in this chain corresponds to a relation from $\mathcal{R} \cup \{(cud, \varepsilon)\}$. Hence the transition $w_{k-1} \to w_k$ can be only the deletion of cud from a subword $ccudd$ of w_{k-1} which leads to some of first three factros cd appearing in w_k. By the same method as in the proof of Proposition 8.3 it is easy to provide a chain from c^2d to $p_1cdp_2cdp_3cdp_4$ of length less than k. $\boxed{12.4}$

In the proof of Proposition 10.2, if $S_{u,v}$ is non-trivial, then $\overline{a}$ does not lie in the same $\mathcal{D}$-class as the identity and so $S_{u,v}$ is bisimple if and only if it is trivial.

Corollary 12.5. *Bisimplicity is undecidable for finitely presented semigroups.*

We do not know whether it is decidable for one-relation monoids to be simple (or bisimple). A one-relation semigroup is simple or bisimple if and only if it is a group:

Corollary 12.6. *It is decidable whether a one-relation semigroup is simple, and whether it is bisimple.*

Proof of 12.6. For a non-trivial semigroup S to be simple (respectively, bisimple), each of its elements must be decomposable. Thus $S^2 = S$ and S is a group by Corollary 5.3. So S is simple (respectively, bisimple) if and only if $S^2 = S$, which is decidable. $\boxed{12.6}$

The question of semisimplicity for one-relation semigroups and monoids remains open. The following is a partial result:

Proposition 12.7. *Let* $S = \mathrm{Sg}\langle A \mid (u, v)\rangle$ *where* $|u|, |v| \geqslant 2$ *and* $|A| \geqslant 2$. *Then* S *is not semisimple.*

Proof of 12.7. Suppose, with the aim of obtaining a contradiction, that S is semisimple. Let $a \in A$. Let $I = S^1\overline{a}S^1$; then I is an ideal of S. Let $J = I^1\overline{a}I^1$; then J is an ideal of I. Therefore J is an ideal of S.

Let $b \in A - \{a\}$. Now, $\overline{a} \in J$ and so $\overline{ab} \in J$ (since J is an ideal). So there are words $p, q \in (A^*aA^*) \cup \{\varepsilon\}$ with $\overline{paq} = \overline{ab}$. Since $q \neq b$, a sequence of transitions must lead from paq to ab. But u and v are of length at least 2, so either $u = ab$ or $v = ab$; assume, without loss of generality $u = ab$.

Similarly, $\overline{ba} \in J$ for any $b \in A$, which forces $v = ba$, and, furthermore, $|A| = 2$ (otherwise S would split into a semigroup free product with one factor being a free semigroup, and so would not be semisimple). Thus

$$S = \mathrm{Sg}\langle A \mid (ab, ba)\rangle = \mathbb{N}_0 \times \mathbb{N}_0 - \{(0, 0)\}.$$

Let $T = \{(x, y) : x \geqslant 2 \wedge y \geqslant 2\}$; then T is an ideal of S. Let $U = T - \{(x, y) : x = 3 \vee y = 3\}$. Then U is an ideal of T but not of S. Therefore S is not semisimple, which is a contradiction; this completes the proof. $\boxed{12.7}$

13 A PROPER DIRECT PRODUCT?

A semigroup S is a proper direct product if $S \simeq T \times U$ for some *non-trivial* semigroups T and U.

Proposition 13.1. *Neither the property of being a proper direct product, nor its negation, is a Markov property.*

Proof of 13.1. Let S be a semigroup. Then S embeds into the direct product $S^1 \times M$, where M is a non-trivial finitely presented monoid. Thus 'being a proper direct product' is not a Markov property.

In addition, S embeds into $S *_S \mathbb{N}$, which is not a proper direct product. Thus 'not being a proper direct product' is not a Markov property. $\boxed{13.1}$

Proposition 13.2. *It is undecidable whether a finitely presented monoid is a proper direct product.*

Proof of 13.2. Take an arbitrary monoid $M = \mathrm{Mon}\langle A \mid \mathcal{R}\rangle$ with insoluble word problem. Pick $u, v \in A^*$ and construct the semigroup

$$\begin{aligned} S'_{u,v} &= \mathrm{Mon}\langle A, c, d, x, y \mid \mathcal{R}, (cud, \varepsilon), (xy, yx), (acvd, cvd)\ \forall a \in A \cup \{c, d\}\rangle \\ &\simeq S_{u,v} *_M (\mathbb{N}_0 \times \mathbb{N}_0) \end{aligned}$$

If $\overline{u} = \overline{v}$ then $S'_{u,v}$ is isomorphic to the direct product $\mathbb{N}_0 \times \mathbb{N}_0$. If $\overline{u} \neq \overline{v}$ then $S'_{u,v}$ is a [monoid] free product of two non-trivial semigroups and so cannot be a proper direct product. $\boxed{13.2}$

Proposition 13.3. *It is decidable whether a one-relation semigroup is a proper direct product.*

Proof of 13.3. Suppose a one-relation semigroup S is a proper direct product $S = U \times V$. Since S is finitely generated [RRW98, Theorem 2.1], this implies that $U^2 = U$ and $V^2 = V$. Thus $S^2 = S$.

So, to decide whether a one-relation semigroup is a proper direct product, first apply Proposition 5.1 to check that $S^2 = S$. By Corollary 5.3, S is a finite cyclic group; whether this is a proper direct product reduces to checking whether its order is other than a power of a prime number. $\boxed{13.3}$

14 A proper free product?

A semigroup S is a proper [semigroup] free product if it is the semigroup free product of some semigroups T and U; it is a proper [monoid] free product if it is the monoid free product of two non-trivial monoids T and U.

Proposition 14.1. *Neither the property of being a proper (semigroup) free product, nor its negation, is a Markov property.*

Proof of 14.1. The proof is analogous to that of Proposition 13.1. $\boxed{14.1}$

The different notions of semigroup and monoid free products require two separate results:

Proposition 14.2. *It is undecidable whether a finitely presented monoid is a proper (monoid) free product.*

Proof of 14.2. Take an arbitrary monoid $S = \mathrm{Mon}\langle A \mid \mathcal{R}\rangle$ with insoluble word problem. Pick $u, v \in A^*$ and construct the semigroup

$$\begin{aligned} S'_{u,v} &= \mathrm{Mon}\big\langle A, c, d, x \mid \mathcal{R}, (cud, \varepsilon), (x^2, x), (acvd, cvd)\ \forall a \in A \cup \{c, d\}\big\rangle \\ &= S_{u,v} *_M \mathrm{Mon}\big\langle x \mid (x^2, x)\big\rangle . \end{aligned}$$

If $\overline{u} = \overline{v}$ then $S'_{u,v} \simeq \mathrm{Mon}\big\langle x \mid x^2 = x\big\rangle$ and so is not a proper free product. If $\overline{u} \neq \overline{v}$ then $S_{u,v}$ is non-trivial and so $S'_{u,v}$ is a proper free product. $\boxed{14.2}$

Proposition 14.3. *It is undecidable whether a finitely presented semigroup is a proper (semigroup) free product.*

Proof of 14.3. Take an arbitrary monoid $S = \mathrm{Mon}\langle A \mid \mathcal{R}\rangle$ with insoluble word problem. Pick $u, v \in A^*$ and let

$$H = S_{u,v} \times \mathrm{Sg}\big\langle e, f \mid (e^2, e), (f^2, f)\big\rangle$$

By [RRW98, Theorem 3.5], H is finitely presented. If $\overline{u} = \overline{v}$ then $S_{u,v}$ is trivial and $H \simeq \mathrm{Sg}\big\langle e, f \mid (e^2, e), (f^2, f)\big\rangle$, which is a proper free product. If $\overline{u} \neq \overline{v}$ then H is a proper direct product and therefore not a proper free product. $\boxed{14.3}$

Proposition 14.4. *It is decidable whether a one-relation semigroup is a proper (semigroup) free product.*

Proof of 14.4. The aim is to prove that $S = \mathrm{Sg}\langle A \mid (u, v)\rangle$ (where $|u| \leqslant |v|$) is a proper free product if and only if one of the following three conditions holds:

1. A contains letters which appear in neither u nor v,
2. $u \in A$, $v \in (A \setminus \{u\})^+$ and $|A| \geqslant 3$,
3. $u = v$ and $|A| \geqslant 2$.

The sufficiency of any one of these three conditions is clear. Assume that $u \neq v$, every letter in A appears in at least one of u and v, and that S is a proper free product $T *_S U$. If $|u|, |v| \geqslant 2$ then every letter from A is indecomposable and so lies in one of the two free factors T and U. Let $u = u'u''$ and $v = v'v''$ where u' and v' are the maximal prefixes of u and v such that $\overline{u'}$ and $\overline{v'}$ lie in the same free factor, which we assume without loss of generality to be T. (Note that u'' and v'' may be empty.) Then, from normal forms for free products, $\overline{u'} = \overline{v'}$ and $\overline{u''} = \overline{v''}$. But no sequence of transitions can lead from u' to v' and u'' to v'' unless these words are equal. So $u = v$, which is a contradiction.

Therefore $u = a$. If v contains the letter a then, again by normal forms for free products, it must coincide with the word a, which is a contradiction. So that $v \in (A \setminus \{a\})^+$ and we are done. $\boxed{14.4}$

The correspondent question on when a one-relation monoid is a proper monoid free product remains open.

15 HOPFIAN?

A semigroup S is *hopfian* if any surjective endomorphism of S is injective and thus an automorphism.

PROPOSITION 15.1. *Neither hopficity nor non-hopficity is a Markov property.*

Proof of 15.1. Consider an arbitrary finitely presented semigroup S. Let T be a non-hopfian semigroup. Then S embeds into $S *_S T$, which is non-hopfian since any non-injective surjection from T onto itself can be extended in a natural way to a non-injective surjection from $S *_S T$ onto itself.

Let S be presented by $\mathrm{Sg}\langle A \mid \mathcal{R}\rangle$, where $A = \{a_1, \ldots, a_n\}$. Let $C = \{c, d\}$ and in each relation from $\mathcal{R}$ replace each letter a_i with cd^i and denote the resulting relations by $\mathcal{R}'$. Let T be the semigroup $\mathrm{Sg}\langle C \mid \mathcal{R}'\rangle$. Notice that, since c and d are indecomposable in T, the only non-trivial surjective endomorphism ϑ of T onto itself could be those which is given by $c \mapsto d$ and $d \mapsto c$, but then ϑ^2 would be the identity mapping and so ϑ is injective. Hence T is hopfian. It remains to note that the subsemigroup in T, generated by $\{cd^i : 1 \leqslant i \leqslant n\}$, is isomorphic to S. $\boxed{15.1}$

PROPOSITION 15.2. *Hopficity is undecidable for finitely presented semigroups.*

Proof of 15.2. Consider the Baumslag-Solitar group $B = \mathrm{Gp}\big\langle x, y \mid (yx^3, x^2y)\big\rangle$. Then the mapping $\vartheta : B \to B$ extending $\overline{x} \mapsto \overline{x^2}$, $\overline{y} \mapsto \overline{y}$ is a surjective endomorphism of B (see [LS77]).

Now consider an arbitrary finitely presented group G with insoluble word problem. Take any finite monoid presentation $\mathrm{Mon}\langle A \mid \mathcal{R}\rangle$ for G. Let $S = G \times B$. Choose $v \in A^*$ and form the monoid $S_{\varepsilon,v}$.

Now, if $\overline{v} = \overline{\varepsilon}$, then $S_{\varepsilon,v}$ is trivial and so hopfian. Suppose that $\overline{v} \neq \overline{\varepsilon}$. Then S embeds into $S_{\varepsilon,v}$. Notice that $S_{\varepsilon,v}$ is generated by $\overline{A} \cup \overline{\{c, d, x, x^{-1}, y, y^{-1}\}}$ and define the mapping $\vartheta^* : S_{\varepsilon,v} \to S_{\varepsilon,v}$ by extending $\vartheta : B \to B$ by setting $\overline{a}\vartheta^* = \overline{a}$ for $a \in A \cup \{c, d\}$. Then ϑ^* is a surjective endomorphism of $S_{\varepsilon,v}$, but ϑ^* is not injective since ϑ is not injective on B. Thus, when $\overline{v} \neq \overline{\varepsilon}$, the semigroup $S_{\varepsilon,v}$ is non-hopfian.

Since the word problem for G is insoluble, hopficity is undecidable for finitely presented semigroups. $\boxed{15.2}$

We conjecture that every one-relation monoid is hopfian.

16 RESIDUALLY FINITE?

PROPOSITION 16.1. *Residual finiteness is a Markov property.*

Proof of 16.1. Let S be a finitely presented semigroup with unsolvable word problem. Then, since finitely presented residually finite semigroups have solvable word problem [Eva69], S cannot embed into a finitely presented residually finite semigroup. $\boxed{16.1}$

For some partial results on the decidability of residual finiteness for one-relation monoids, see [Lal74].

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