An Annotated Mathematician's Apology

G. H. HARDY

An Annotated Mathematician's Apology

Annotations and commentary by Alan J. Cain

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ANNOTATOR'S PREFACE

Although G. H. Hardy, in his mathematical writing, was 'above the average in his care to cite others and provide bibliographies in his books',[1] *A Mathematician's Apology* is filled with quotations, allusions, and references that are often unsourced.

This annotated edition aims to give sources for all quotations and clarify allusions to works, people, or events, as well as adding background information. Hardy made a number of minor misquotations, suggesting that he quoted from memory or used paraphrased notes of his own; the annotations point these out. This edition also includes an annotated version of Hardy's essay 'Mathematics in war-time', which formed the kernel around which Hardy shaped the *Apology*. The annotations point out how parts of this essay were incorporated into the *Apology*.

In both the *Apology* and 'Mathematics in war-time', Hardy's original footnotes are preserved and marked with an asterisk * or a dagger † . The annotations are in numbered footnotes.

Also included is a list of editions, excerptings, and translations of the *Apology* and 'Mathematics in war-time', and three essays by the annotator: the first sets the *Apology* in context in the debate about the justification for mathematics, particularly as an aesthetic

1 Grattan-Guinness, 'The interest of G.H. Hardy', p. 412.

pursuit; the second attempts to survey comprehensively contemporary reviews of the *Apology*; the third examines the legacy and ongoing influence of the *Apology*.

This annotated edition of *A Mathematician's Apology* is a 'beta version'. The annotator welcomes any comments, corrections, or constructive criticisms. Particularly welcome is information about editions, excerptings, or translations of the *Apology* or 'Mathematics in war-time' other than those listed on pp. 69 sqq.; information about contemporary reviews other than those considered on pp. 97 sqq.; or copies of the various reviews that the annotator has been unable to obtain (see pp. 101, 102, 103, 104).

The annotator thanks Yumi Murayama for reading and commenting on this edition, and Erkko Lehtonen for supplying details of the Finnish translation and for pointing out typos.

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Lisbon,
21 *January* 2019

A. J. C.

A
Mathematician's
Apology

To

JOHN LOMAS[1]

who asked me to write it

1 John Millington Lomas (1917–45): cricketer; fellow of New College, Oxford; close friend of Hardy.

PREFACE

I am indebted for many valuable criticisms to Professor C. D. Broad[2] and Dr C. P. Snow,[3] each of whom read my original manuscript. I have incorporated the substance of nearly all of their suggestions in my text, and have so removed a good many crudities and obscurities.

In one case I have dealt with them differently. My § 28 is based on a short article[4] which I contributed to *Eureka* (the journal of the

2 Charlie Dunbar Broad (1887–1971): philosopher and historian of philosophy; fellow of Trinity College, Cambridge. Broad wrote a positive (though not uncritical) review of the *Apology* for the journal *Philosophy*; see pp. 99 sq..

3 Charles Percy Snow, Baron Snow (1905–80): chemist, novelist, and civil servant; fellow of Christ's College, Cambridge; friend of Hardy. Snow was noted for his lecture *The Two Cultures* on the division between the sciences and the humanities. He wrote a biographical study of Hardy that related his own memories of their friendship. This essay first appeared in *The Atlantic Monthly* (Snow, 'G.H. Hardy: the pure mathematician'), was reprinted as one of nine biographies in Snow's book *Variety of Men*, and was used as the foreword to the 1967 and subsequent reprintings of the *Apology*.

4 Hardy, 'Mathematics in war-time'. Parts of this article are actually found in § 21, § 25, and § 28. The annotations to the reprinting of the article on pp. 64 sqq. give

Cambridge Archimedean Society[5]) early in the year, and I found it impossible to remodel what I had written so recently and with so much care. Also, if I had tried to meet such important criticisms seriously, I should have had to expand this section so much as to destroy the whole balance of my essay. I have therefore left it unaltered, but have added a short statement of the chief points made by my critics in a note at the end.

G. H. H.

18 *July* 1940

1

It is a melancholy experience for a professional mathematician to find himself writing about mathematics. The function of a mathematician is to do something, to prove new theorems, to add to mathematics, and not to talk about what he or other mathematicians have done. Statesmen despise publicists, painters despise art-critics, and physiologists, physicists, or mathematicians have usually similar feelings; there is no scorn more profound, or on the whole more justifiable, than that of the men who make for the men who explain. Exposition, criticism, appreciation, is work for second-rate minds.

I can remember arguing this point once in one of the few serious conversations that I ever had with Housman.[6] Housman, in his Leslie Stephen[7] lecture *The Name and Nature of Poetry*, had denied very emphatically that he was a 'critic'; but he had denied it in what seemed to me a singularly perverse way, and had expressed an admiration for literary criticism which startled and scandalized

details of its incorporation into the *Apology*.

5 The University of Cambridge undergraduate mathematical society, founded in 1935.

6 Alfred Edward Housman (1859–1936): classicist and poet; Professor of Latin and fellow of Trinity College, Cambridge.

7 Named for Leslie Stephen (1832–1904): biographer, historian, and critic.

me. He had begun with a quotation from his inaugural lecture, delivered twenty-two years before —

> 'Whether the faculty of literary criticism is the best gift that Heaven has in its treasuries, I cannot say; but Heaven seems to think so, for assuredly it is the gift most charily bestowed. Orators and poets…, if rare in comparison with blackberries, are commoner than returns of Halley's comet: literary critics are less common…'[8]

And he had continued —

> 'In these twenty-two years I have improved in some respects and deteriorated in others, but I have not so much improved as to become a literary critic, nor so much deteriorated as to fancy that I have become one.'[9]

It had seemed to me deplorable that a great scholar and a fine poet should write like this, and, finding myself next to him in Hall a few weeks later, I plunged in and said so. Did he really mean what he had said to be taken very seriously? Would the life of the best of critics really have seemed to him comparable with that of a scholar and a poet? We argued these questions all through dinner, and I think that finally he agreed with me. I must not seem to claim a dialectical triumph over a man who can no longer contradict me;[10] but 'Perhaps not entirely' was, in the end, his reply to the first question, and 'Probably no' to the second. There may have been some doubt about Housman's feelings, and I do not wish to claim him as on my side; but there is no doubt at all about the feelings of men of science, and I share them fully. If then I find myself writing, not mathematics but 'about' mathematics, it is a confession of weakness, for which I may rightly be scorned or pitied by younger and more vigorous mathematicians. I write about mathematics because, like any other mathematician who has passed sixty, I have no longer the freshness of mind, the energy, or the patience to carry on effectively with my proper job.

8 Housman, *The Name and Nature of Poetry*, p. 5.

9 *Ibid.*, p. 6.

10 The *Apology* was published in 1940, four years after Housman's death.

2

I propose to put forward an apology for mathematics; and I may be told that it needs none, since there are now few studies more generally recognized, for good reasons or bad, as profitable and praiseworthy. This may be true; indeed it is probable, since the sensational triumphs of Einstein,[11] that stellar astronomy and atomic physics are the only sciences which stand higher in popular estimation. A mathematician need not now consider himself on the defensive. He does not have to meet the sort of opposition described by Bradley[12] in the admirable defence of metaphysics which forms the introduction to *Appearance and Reality*.

A metaphysician, says Bradley, will be told that 'metaphysical knowledge is wholly impossible',[13] or that 'even if possible to a certain degree, it is practically no knowledge worth the name'.[14] 'The same problems,' he will hear, 'the same disputes, the same sheer failure. Why not abandon it and come out? Is there nothing else more worth your labour?'[15] There is no one so stupid as to use this sort of language about mathematics. The mass of mathematical truth is obvious and imposing; its practical applications, the bridges and steam-engines and dynamos, obtrude themselves on the dullest imagination. The public does not need to be convinced that there is something in mathematics.

All this is in its way very comforting to mathematicians, but it is hardly possible for a genuine mathematician to be content with it. Any genuine mathematician must feel that it is not on these crude achievements that the real case for mathematics rests, that the popular reputation of mathematics is based largely on ignorance and confusion, and that there is room for a more rational defence.

11 Albert Einstein (1879–1955): physicist; 1921 Nobel laureate in physics.

12 Francis Herbert Bradley (1846–1924): British idealist philosopher.

13 Bradley, *Appearance and Reality*, p. 1.

14 Hardy misquotes Bradley here. The original text (in both editions) reads: '[it] may be possible theoretically, and even actual, if you please, to a certain degree; but, for all that, it is practically no knowledge worth the name' (*ibid.*, p. 2).

15 *Loc. cit.*

At any rate, I am disposed to try to make one. It should be a simpler task than Bradley's difficult apology.

I shall ask, then, why is it really worth while to make a serious study of mathematics? What is the proper justification of a mathematician's life? And my answers will be, for the most part, such as are to be expected from a mathematician: I think that it is worth while, that there is ample justification. But I should say at once that my defence of mathematics will be a defence of myself, and that my apology is bound to be to some extent egotistical. I should not think it worth while to apologize for my subject if I regarded myself as one of its failures.

Some egotism of this sort is inevitable, and I do not feel that it really needs justification. Good work is not done by 'humble' men. It is one of the first duties of a professor, for example, in any subject, to exaggerate a little both the importance of his subject and his own importance in it. A man who is always asking 'Is what I do worth while?' and 'Am I the right person to do it?' will always be ineffective himself and a discouragement to others. He must shut his eyes a little and think a little more of his subject and himself than they deserve. This is not too difficult: it is harder not to make his subject and himself ridiculous by shutting his eyes too tightly.

3

A man who sets out to justify his existence and his activities has to distinguish two different questions. The first is whether the work which he does is worth doing; and the second is why he does it, whatever its value may be. The first question is often very difficult, and the answer very discouraging, but most people will find the second easy enough even then. Their answers, if they are honest, will usually take one or other of two forms; and the second form is merely a humbler variation of the first, which is the only answer which we need consider seriously.

(1) 'I do what I do because it is the one and only thing that I

can do at all well. I am a lawyer, or a stockbroker, or a professional cricketer, because I have some real talent for that particular job. I am a lawyer because I have a fluent tongue, and am interested in legal subtleties; I am a stockbroker because my judgement of the markets is quick and sound; I am a professional cricketer because I can bat unusually well. I agree that it might be better to be a poet or a mathematician, but unfortunately I have no talent for such pursuits.'

I am not suggesting that this is a defence which can be made by most people, since most people can do nothing at all well. But it is impregnable when it can be made without absurdity, as it can by a substantial minority: perhaps five or even ten per cent of men can do something rather well. It is a tiny minority who can do anything really well, and the number of men who can do two things well is negligible. If a man has any genuine talent, he should be ready to make almost any sacrifice in order to cultivate it to the full.

This view was endorsed by Dr Johnson[16] —

> 'When I told him that I had been to see [his namesake] Johnson ride upon three horses, he said "Such a man, sir, should be encouraged, for his performances show the extent of the human powers…"' — [17]

and similarly he would have applauded mountain climbers, channel swimmers, and blindfold chess-players. For my own part, I am entirely in sympathy with all such attempts at remarkable achievement. I feel some sympathy even with conjurors and ventriloquists; and when Alekhine[18] and Bradman[19] set out to beat records, I am quite bitterly disappointed if they fail. And here both Dr Johnson

16 Samuel Johnson (1709–84): poet, author, and lexicographer; author of *A Dictionary of the English Language* (1755).

17 Boswell, *The Life of Samuel Johnson, LL.D.* vol. 1, p. 215. The quotation actually begins mid-sentence. Hardy seems to quote from a later edition with updated spelling, capitalization, and punctuation.

18 Alexander Alekhine [Александр Александрович Алехин] (1892–1946): chess player, widely regarded as one of the greatest ever.

19 Donald George Bradman (1908–2001): cricketer, widely regarded as the greatest ever batsman.

and I find ourselves in agreement with the public. As W. J. Turner[20] has said so truly, it is only the 'highbrows' (in the unpleasant sense) who do not admire the 'real swells'.[21]

We have of course to take account of the differences in value between different activities. I would rather be a novelist or a painter than a statesman of similar rank; and there are many roads to fame which most of us would reject as actively pernicious. Yet it is seldom that such differences of value will turn the scale in a man's choice of a career, which will almost always be dictated by the limitations of his natural abilities. Poetry is more valuable than cricket, but Bradman would be a fool if he sacrificed his cricket in order to write second-rate minor poetry (and I suppose that it is unlikely that he could do better). If the cricket were a little less supreme, and the poetry better, then the choice might be more difficult: I do not know whether I would rather have been Victor Trumper[22] or Rupert Brooke.[23] It is fortunate that such dilemmas occur so seldom.

I may add that they are particularly unlikely to present themselves to a mathematician. It is usual to exaggerate rather grossly the differences between the mental processes of mathematicians and other people, but it is undeniable that a gift for mathematics is one of the most specialized talents, and that mathematicians as a class are not particularly distinguished for general ability or versatility. If a man is in any sense a real mathematician, then it is a hundred to one that his mathematics will be far better than anything else he can do, and that he would be silly if he surrendered any decent opportunity of exercising his one talent in order to do undistinguished work in other fields.[24] Such a sacrifice could be

20 Walter James Redfern Turner (1889–1946): poet, playwright, critic.

21 See the discussion in Turner, *Variations on the Theme of Music*.

22 Victor Thomas Trumper (1877–1915): cricketer, acclaimed as a great batsman.

23 Rupert Chawner Brooke (1887–1915): poet and writer.

24 According to his friend C. P. Snow, 'Hardy in his secret heart really felt that anyone ought to do pure mathematics if he had the talent for it' (Snow, 'The Classical Mind', p. 813), and this gave an edge to his admiration of Dirac as a theoretical physicist, for Hardy felt that Dirac could have excelled as a pure mathematician. Hardy still counted Dirac as a 'real mathematician' (see § 25).

justified only by economic necessity or age.

4

I had better say something here about this question of age, since it is particularly important for mathematicians. No mathematician should ever allow himself to forget that mathematics, more than any other art or science, is a young man's game.[25] To take a simple illustration at a comparatively humble level, the average age of election to the Royal Society is lowest in mathematics.[26]

We can naturally find much more striking illustrations. We may consider, for example, the career of a man who was certainly

25 There is, however, evidence that outstanding creativity decreases with age in other sciences as well. This point had been made by contemporaries of Hardy. See Lehman, 'The age decrement in outstanding scientific creativity' and in particular its last section. Krebs, 'Comments on the productivity of scientists' says that it is less pronounced in other sciences.

On the other hand, statistical studies using publication and citation counts have found no reduction in productivity with age among mathematicians (Stern, 'Age and Achievement in Mathematics'; Cole, 'Age and Scientific Performance').

Hardy made the same point about age in his lectures inspired by Ramanujan's work:

> 'a mathematician is often comparatively old at thirty, and his death may be less of a catastrophe than it seems. Abel died at twenty-six and, although he would no doubt have added a great deal more to mathematics, he could hardly have become a greater man.' (Hardy, *Ramanujan*, p. 6)

26 It is effectively impossible to judge whether this statement is correct.Hardy's view of 'real' mathematicians (see § 25) includes certain mathematical physicists; more generally, it depends on how one divides the sciences. However, the evidence does not seem to *contradict* Hardy's statement.

Hardy was elected a fellow of the Royal Society in 1910 and from this date could nominate and vote for new fellows. The *Apology* was published in 1940. During the thirty year period 1911–40, there were 480 fellows elected, not including Foreign Members, 'Statute 12 Fellows' (effectively honorary fellows), or 'Royal Fellows'. The annotator, making a very subjective allocation of these fellows to the various sciences, and considering only fields with at least 10 fellows, obtained the following results:

one of the world's three greatest mathematicians.[27] Newton[28] gave up mathematics at fifty, and had lost his enthusiasm long before; he had recognized no doubt by the time that he was forty that his great creative days were over. His greatest ideas of all, fluxions and the law of gravitation, came to him about 1666, when he was twenty-four — 'in those days I was in the prime of my age for invention, and minded mathematics and philosophy more than at any time since'.[29] He made big discoveries until he was nearly forty (the 'elliptic orbit' at thirty-seven), but after that he did little but polish and perfect.

Galois[30] died at twenty-one, Abel[31] at twenty-seven, Ramanu-

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	Age at election	
Fields	Mean	Median
Mathematics and physics	42	40
Chemistry	46	45
Life sciences and medicine	48	47
Geology	53	52
Engineering	57	54

Mathematics and physics are grouped together because of Hardy's view of 'real' mathematics; the annotator declines to judge how Hardy would divide these fellows.

27 Hardy agreed with the traditional view (see, for example, Klein, *Geometry*, p. 215 [206] or Bell, *Men of Mathematics*, chs 2, 14) that Archimedes, Newton, and Gauss were the three greatest mathematicians in history. Hardy classified mathematicians by comparison to cricketers, and, writing to C. P. Snow, placed these three mathematicians in the 'Bradman' class (see § 3), saying that 'Bradman is a whole class above any batsman who has ever lived' (Snow, 'Foreword', p. 28).

28 Isaac Newton (1642–1726 OS/27 NS): mathematician, astronomer, and natural philosopher; along with Leibniz, creator of the calculus.

29 University Library, Cambridge, Additional Manuscript 3968.41, f. 85r. The quotation, as given in Newton, *Mathematical Papers 1664–1666*, p. 152, is: 'in those days I was in the prime of my age for invention & minded Mathematicks & Philosophy more then at any time since.'

30 Évariste Galois (1811–32): mathematician; determined conditions for the solubility by radicals of polynomial equations.

31 Niels Henrik Abel (1802–29): mathematician; proved that quintic equation is not soluble by radicals.

jan[32] at thirty-three, Riemann[33] at forty. There have been men who have done great work a good deal later; Gauss's[34] great memoir on differential geometry was published when he was fifty (though he had had the fundamental ideas ten years before). I do not know an instance of a major mathematical advance initiated by a man past fifty.[35] If a man of mature age loses interest in and abandons mathematics, the loss is not likely to be very serious either for mathematics or for himself.

On the other hand the gain is no more likely to be substantial; the later records of mathematicians who have left mathematics are not particularly encouraging. Newton made a quite competent Master of the Mint[36] (when he was not quarrelling with anybody). Painlevé[37] was a not very successful Premier of France. Laplace's[38] political career was highly discreditable, but he is hardly a fair instance, since he was dishonest rather than incompetent, and never really 'gave up' mathematics. It is very hard to find an instance of a first-rate mathematician who has abandoned mathematics and attained first-rate distinction in any other field.[*] There may have been young men who would have been first-rate mathematicians if they had stuck to mathematics, but I have never heard

32 Srinivasa Ramanujan [ஸ்ரீனிவாச ராமானுஜன்] (1887–1920): mathematician; major contributions to number theory and combinatorics.

33 Georg Friedrich Bernhard Riemann (1826–66): mathematician; major contributions to analysis and number theory.

34 Johann Carl Friedrich Gauss [Gauß] (1777–1855): prolific mathematician; major contributions to many fields.

35 Laplace, whom Hardy mentioned in the next paragraph, made major contributions to probability and mathematical physics well into his sixties.

36 The official who oversaw the Royal Mint, which produced the coinage of England and later the United Kingdom of Great Britain. Newton was Master of the Mint from 1700 until his death.

37 Paul Painlevé (1863–1933): mathematician and politician; Prime Minister of the French Third Republic in 1917 and again in 1925.

38 Pierre-Simon Laplace (1749–1827): mathematician, physicist, and astronomer.

* Pascal[39] seems the best.

39 Blaise Pascal (1623–62): mathematician, physicist, and theologian. After a religious experience in 1654, Pascal abandoned mathematics for theology and philosophy.

of a really plausible example. And all this is fully borne out by my own very limited experience. Every young mathematician of real talent whom I have known has been faithful to mathematics, and not from lack of ambition but from abundance of it; they have all recognized that there, if anywhere, lay the road to a life of any distinction.

5

There is also what I called the 'humbler variation' of the standard apology; but I may dismiss this in a very few words.

(2) 'There is nothing that I can do particularly well. I do what I do because it came my way. I really never had a chance of doing anything else.' And this apology too I accept as conclusive. It is quite true that most people can do nothing well. If so, it matters very little what career they choose, and there is really nothing more to say about it. It is a conclusive reply, but hardly one likely to be made by a man with any pride; and I may assume that none of us would be content with it.

6

It is time to begin thinking about the first question which I put in § 3, and which is so much more difficult than the second. Is mathematics, what I and other mathematicians mean by mathematics, worth doing; and if so, why?

I have been looking again at the first pages of the inaugural lecture which I gave at Oxford in 1920,[40] where there is an outline

40 Hardy, *Some Famous Problems*. This inaugural lecture appears to employ a trenchant irony not present in the *Apology* or 'Mathematics in war-time' when Hardy discusses the usefulness of mathematics:

> 'I must leave it to the engineers and the chemists to expound, with

of an apology for mathematics. It is very inadequate (less than a couple of pages), and it is written in a style (a first essay, I suppose, in what I then imagined to be the 'Oxford manner') of which I am not now particularly proud; but I still feel that, however much development it may need, it contains the essentials of the matter. I will resume what I said then, as a preface to a fuller discussion.

(1) I began by laying stress on the *harmlessness* of mathematics — 'the study of mathematics is, if an unprofitable, a perfectly harmless and innocent occupation.'[41] I shall stick to that, but obviously it will need a good deal of expansion and explanation.

Is mathematics 'unprofitable'? In some ways, plainly, it is not; for example, it gives great pleasure to quite a large number of people. I was thinking of 'profit', however, in a narrower sense. Is mathematics 'useful', *directly* useful, as other sciences such as chemistry and physiology are? This is not an altogether easy or uncontroversial question, and I shall ultimately say No, though some mathematicians, and most outsiders, would no doubt say Yes. And is mathematics 'harmless'? Again the answer is not obvious, and the question is one which I should have in some ways preferred to avoid, since it raises the whole problem of the effect of science on war. Is mathematics harmless, in the sense in which, for example, chemistry plainly is not? I shall have to come back to both these questions later.

(2) I went on to say that 'the scale of the universe is large and, if we are wasting our time, the waste of the lives of a few university

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> justly prophetic fervour, the benefits conferred on civilization by gas-engines, oil, and explosives. If I could attain every scientific ambition of my life, the frontiers of the Empire would not be advanced, not even a black man would be blown to pieces, no one's fortune would be made, and least of all my own. A pure mathematician must leave to happier colleagues the great task of alleviating the sufferings of humanity.' (Hardy, *Some Famous Problems*, p. 4)

The ironic expression of pacificism is marred by the apparently unironic racism. The editors of the generally admirable anthology *The G.H. Hardy Reader* bowdlerized Hardy by eliding from 'not even' to the end of the sentence (Albers, Alexanderson & Dunham, *The G.H. Hardy Reader*, p. 371).

41 Hardy, *Some Famous Problems*, p. 4.

dons is no such overwhelming catastrophe':[42] and here I may seem to be adopting, or affecting, the pose of exaggerated humility which I repudiated a moment ago. I am sure that that was not what was really in my mind; I was trying to say in a sentence what I have said at much greater length in § 3. I was assuming that we dons really had our little talents, and that we could hardly be wrong if we did our best to cultivate them fully.

(3) Finally (in what seem to me now some rather painfully rhetorical sentences) I emphasized the permanence of mathematical achievement —

> 'What we do may be small, but it has a certain character of permanence; and to have produced anything of the slightest permanent interest, whether it be a copy of verses or a geometrical theorem, is to have done something utterly beyond the powers of the vast majority of men.'[43]

And —

> 'In these days of conflict between ancient and modern studies, there must surely be something to be said for a study which did not begin with Pythagoras,[44] and will not end with Einstein, but is the oldest and the youngest of all.'[45]

All this is 'rhetoric'; but the substance of it seems to me still to ring true, and I can expand it at once without prejudging any of the other questions which I am leaving open.

42 Hardy, *Some Famous Problems*, p. 4. Hardy here omitted a comma after 'large'.

43 *Ibid.*, pp. 4–5. The quotation begins mid-sentence.

44 Pythagoras of Samos [Πυθαγόρας *Pythagóras*] (c. 570–c. 495 BCE): religious leader; possibly a philosopher and mathematician. For a survey of the debate on whether Pythagoras contributed to philosophy and mathematics, see Kahn, *Pythagoras and the Pythagoreans*, Preface.

45 Hardy, *Some Famous Problems*, p. 5. Again, the quotation begins mid-sentence.

7

I shall assume that I am writing for readers who are full, or have in the past been full, of a proper spirit of ambition. A man's first duty, a young man's at any rate, is to be ambitious. Ambition is a noble passion which may legitimately take many forms; there was *something* noble in the ambition of Attila[46] or Napoleon:[47] but the noblest ambition is that of leaving behind one something of permanent value —

> 'Here, on the level sand,
> Between the sea and land,
> What shall I build or write
> Against the fall of night?
>
> Tell me of runes to grave
> That hold the bursting wave,
> Or bastions to design
> For longer date than mine.'[48]

Ambition has been the driving force behind nearly all the best work of the world. In particular, practically all substantial contributions to human happiness have been made by ambitious men. To take two famous examples, were not Lister,[49] and Pasteur[50] ambitious? Or, on a humbler level, King Gillette[51] and William Willett;[52] and who in recent times have contributed more to human comfort than they?

46 Attila (c. 406–453): ruler of the Huns; invader of the Western and Eastern Roman Empires.

47 Napoléon Bonaparte (1769–1821): Emperor of France and conqueror of a large part of Europe.

48 Housman, *More Poems*, 'Smooth Between Sea and Land' (XLV), ll. 9–16.

49 Joseph Lister (1827–1912): surgeon; pioneer of using antiseptic in surgery.

50 Louis Pasteur (1822–95): biologist and chemist; developer of vaccination and pasteurization.

51 King Camp Gillette (1855–1932): inventor of the affordable disposable safety razor.

52 William Willett (1856–1915): campaigner for daylight saving time.

Physiology provides particularly good examples, just because it is so obviously a 'beneficial' study. We must guard against a fallacy common among apologists of science, the fallacy of supposing that the men whose work most benefits humanity are thinking much of that while they do it, that physiologists, for example, have particularly noble souls. A physiologist may indeed be glad to remember that his work will benefit mankind, but the motives which provide the force and the inspiration for it are indistinguishable from those of a classical scholar or a mathematician.

There are many highly respectable motives which may lead men to prosecute research, but three which are much more important than the rest. The first (without which the rest must come to nothing) is intellectual curiosity, desire to know the truth. Then, professional pride, anxiety to be satisfied with one's performance, the shame that overcomes any self-respecting craftsman when his work is unworthy of his talent. Finally, ambition, desire for reputation, and the position, even the power or the money, which it brings. It may be fine to feel, when you have done your work, that you have added to the happiness or alleviated the sufferings of others, but that will not be why you did it. So if a mathematician, or a chemist, or even a physiologist, were to tell me that the driving force in his work had been the desire to benefit humanity, then I should not believe him (nor should I think the better of him if I did). His dominant motives have been those which I have stated, and in which, surely, there is nothing of which any decent man need be ashamed.

8

If intellectual curiosity, professional pride, and ambition are the dominant incentives to research, then assuredly no one has a fairer chance of gratifying them than a mathematician. His subject is the most curious of all — there is none in which truth plays such odd pranks. It has the most elaborate and the most fascinating technique, and gives unrivalled openings for the display of sheer professional skill. Finally, as history proves abundantly, mathematical achievement, whatever its intrinsic worth, is the most enduring of all.

We can see this even in semi-historic civilizations. The Babylonian and Assyrian civilizations have perished; Hammurabi,[53] Sargon,[54] and Nebuchadnezzar[55] are empty names; yet Babylonian mathematics is still interesting, and the Babylonian scale of 60 is still used in astronomy. But of course the crucial case is that of the Greeks.

The Greeks were the first mathematicians who are still 'real' to us to-day. Oriental mathematics may be an interesting curiosity, but Greek mathematics is the real thing. The Greeks first spoke a language which modern mathematicians can understand; as Littlewood[56] said to me once, they are not clever schoolboys or 'scholarship candidates', but 'Fellows of another college'. So Greek mathematics is 'permanent', more permanent even than Greek literature. Archimedes[57] will be remembered when Aeschylus[58] is forgotten, because languages die and mathematical ideas do not. 'Immortality' may be a silly word, but probably a mathematician has the best chance of whatever it may mean.[59]

53 Hammurabi [𒄩𒄠𒈬𒊏𒁉 *Ha-am-mu-ra-bi*] (c. 1810–c. 1750 BCE): King of Babylon and conqueror of Mesopotamia; promulagator of the earliest known code of laws.

54 Sargon of Akkad [𒊬𒊒𒄀 *Šarru-ukīn*] (d. c. 2284 BCE): ruler of the Akkadian Empire; conqueror of Sumer.

55 Nebuchadnezzar II [𒀭𒀝𒆪𒁺𒌨𒊑𒋀 *Nabû-kudurri-uṣur*] (c. 634–c. 562 BCE): longest-reigning King of Babylon.

56 John Edensor Littlewood (1885–1977): number theorist and analyst; long-time collaborator of Hardy.

57 Archimedes of Syracuse [Ἀρχιμήδης *Archimēdēs*] (c. 287–c. 212 BCE): mathematician, engineer, and astronomer; sometimes considered the greatest mathematician ever.

58 Aeschylus [Αἰσχύλος *Aiskhulos*] (c. 525/4–c. 456/5 BCE): tragic dramatist.

59 Hardy made a similar observation in the 1920 inaugural lecture he discussed in § 6: 'The mathematicians of the past have not been neglected or despised; they have been rewarded in a manner, undiscriminating perhaps, but certainly not ungenerous' (Hardy, *Some Famous Problems*, p. 5). The poet, perhaps naturally,

Nor need he fear very seriously that the future will be unjust to him. Immortality is often ridiculous or cruel: few of us would have chosen to be Og[60] or Ananias[61] or Gallio.[62] Even in mathematics, history sometimes plays strange tricks; Rolle[63] figures in the text-books of elementary calculus as if he had been a mathematician like Newton; Farey[64] is immortal because he failed to understand a

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can be equally optimistic of his own 'immortality':

> 'Exegi monumentum aere perennius
> regalique situ pyramidum altius,
> quod non imber edax, non Aquilo impotens
> possit diruere aut innumerabilis
> annorum series et fuga temporum.
> non omnis moriar' (Horace, *Odes*, III.30)

['I have finished a monument more lasting than bronze, more lofty than the regal structure of the pyramids, one which neither corroding rain nor the ungovernable North Wind can ever destroy, nor the countless series of the years, nor the flight of time. I shall not wholly die'.]

60 According to the Bible, the Israelites defeated and killed Og [עוֹג], the king of Bashan, and subsequently destroyed his kingdom and its inhabitants (Numbers 21:33–5; Deuteronomy 3:1–4).

61 Several Biblical figures have this name, but Hardy probably refers to Ananias son of Nebedeus, the high priest who presided over the trials of Paul of Tarsus (Acts 23–4) and who was subsequently killed at the start of the First Jewish-Roman War for being sympathetic to Rome (Josephus, *The Jewish War*, § II.xvii.9). A less likely possibility is that he meant the Ananias who was miraculously punished with sudden death for lying to Peter (and, by extension, God), but this story is usually referred to as 'Ananias and Sapphira', since his wife Sapphira was similarly killed for the same reason soon afterwards (Acts 5:1–11).

62 Lucius Junius Gallio Annaeanus (c. 5 BCE–c. 65 CE): Roman senator, brother of the philosopher Seneca. According to the Bible, Gallio dismissed charges made against Paul of Tarsus by the Jews (Acts 18:12–7).

It is perhaps noteworthy that Hardy, an avowed atheist, chose Biblical figures to illustrate the capriciousness of historical memory.

63 Michel Rolle (1652–1719): mathematician, known for 'Rolle's theorem'. Rolle proved only a special case of this result, but it became one of the fundamental results of real analysis, suggesting that Rolle played an important part in its foundation.

64 John Farey (1766–1826): geologist. Farey, 'On a curious property of vulgar fractions' stated, without proof, some remarks on what are now called *Farey sequences*. They had first been studied by Haros, 'Tables pour évaluer une fraction ordinaire'. Hardy was rather harsh towards Farey, especially since there is no indication that Farey was aware of the work of Haros. Writing with Wright in *An Introduction*

theorem which Haros[65] had proved perfectly fourteen years before; the names of five worthy Norwegians still stand in Abel's *Life*, just for one act of conscientious imbecility, dutifully performed at the expense of their country's greatest man.[66] But on the whole the history of science is fair, and this is particularly true in mathematics. No other subject has such clear-cut or unanimously accepted standards, and the men who are remembered are almost always the men who merit it. Mathematical fame, if you have the cash to pay for it, is one of the soundest and steadiest of investments.

9

All this is very comforting for dons, and especially for professors of mathematics. It is sometimes suggested, by lawyers or politicians or business men, that an academic career is one sought mainly by cautious and unambitious persons who care primarily for comfort and security. The reproach is quite misplaced. A don surrenders something, and in particular the chance of making large sums of money — it is very hard for a professor to make £ 2000 a year; and security of tenure is naturally one of the considerations which make this particular surrender easy. That is not

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to the Theory of Numbers, he said (p. 37):

> 'Farey has a notice of twenty lines in the *Dictionary of National Biography*, where he is described as a geologist. As a geologist he is forgotten, and his biographer does not mention the one thing in his life which survives.'

This remark survived until the fourth edition (1960) but was removed by Wright for the fifth edition (1978).

65 Charles Haros (*fl.* 1801–06): mathematician and civil servant.

66 This presumably refers to the difficulties Abel had in obtaining financial support after returning to Norway from his tour of European mathematical centres in 1825–7 (see Stubhaug, *Niels Henrik Abel and his Times*, chs 44–5), but it is unclear which five individuals Hardy had in mind.

why Housman would have refused to be Lord Simon[67] or Lord Beaverbrook.[68] He would have rejected their careers because of his ambition, because he would have scorned to be a man to be forgotten in twenty years.

Yet how painful it is to feel that, with all these advantages, one may fail. I can remember Bertrand Russell[69] telling me of a horrible dream. He was in the top floor of the University Library, about A.D. 2100. A library assistant was going round the shelves carrying an enormous bucket, taking down book after book, glancing at them, restoring them to the shelves or dumping them into the bucket. At last he came to three large volumes which Russell could recognize as the last surviving copy of *Principia mathematica*. He took down one of the volumes, turned over a few pages, seemed puzzled for a moment by the curious symbolism, closed the volume, balanced it in his hand and hesitated….

10

A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with *ideas*. A painter makes patterns with shapes and colours, a poet with words. A painting may embody an 'idea', but the idea is usually commonplace and unimportant. In poetry, ideas count for a good deal more; but, as Housman insisted, the importance of ideas in poetry is habitually exaggerated:

67 John Allsebrook Simon, 1st Viscount Simon (1873–1954): politician. Simon held several cabinet posts. In particular, he was Chancellor of the Exchequer until May 1940 and subsequently Lord Chancellor.

68 William Maxwell Aitken, 1st Baron Beaverbrook (1879–1964): newspaper publisher and politician.

69 Bertrand Arthur William Russell, 3rd Earl Russell (1872–1970): philosopher, mathematician, social critic; 1950 Nobel laureate in literature. Together with Whitehead, he wrote the *Principia Mathematica*, which endeavoured to establish the foundations of mathematics using symbolic logic.

'I cannot satisfy myself that there are any such things as poetical ideas.... Poetry is not the thing said but a way of saying it.'[70]

> 'Not all the water in the rough rude sea
> Can wash the balm from an anointed King.'[71]

Could lines be better, and could ideas be at once more trite and more false? The poverty of the ideas seems hardly to affect the beauty of the verbal pattern. A mathematician, on the other hand, has no material to work with but ideas, and so his patterns are likely to last longer, since ideas wear less with time than words.

The mathematician's patterns, like the painter's or the poet's, must be *beautiful*; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.[72] And here I must deal with a misconception which is still widespread (though probably much less so now than it was twenty years ago), what Whitehead[73] has called the 'literary superstition'[74] that love of and aesthetic appreciation[75] of mathematics is 'a monomania confined to a few eccentrics in each generation'.[76]

It would be difficult now to find an educated man quite insensitive to the aesthetic appeal of mathematics. It may be very hard to *define* mathematical beauty, but that is just as true of beauty of any kind — we may not know quite what we mean by a beautiful poem, but that does not prevent us from recognizing one when we read it. Even Professor Hogben,[77] who is out to minimize at all

70 Housman, *The Name and Nature of Poetry*, pp. 36 & 37.

71 Shakespeare, *Richard II*, Act 3, Scene ii.

72 The only examples Hardy gave of ugly mathematics are ballistics and aerodynamics (see § 28).

73 Alfred North Whitehead (1861–1947): mathematician and philosopher.

74 Whitehead, *Science and the Modern World*, p. 21, but Whitehead actually calls it 'an erroneous literary tradition'.

75 Hardy seemed to identify aesthetic value with beauty. He did not consider elegance, for example, to be a separate aesthetic value.

76 Whitehead, *Science and the Modern World*, p. 21.

77 Lancelot Thomas Hogben (1895–1975): experimental zoologist and writer of popular science and mathematics books. Hogben is probably most noted in mathematics for his *Mathematics for the Million*.

costs the importance of the aesthetic element in mathematics, does not venture to deny its reality. 'There are, to be sure, individuals for whom mathematics exercises a coldly impersonal attraction.... The aesthetic appeal of mathematics may be very real for a chosen few.'[78] But they are 'few', he suggests, and they feel 'coldly' (and are really rather ridiculous people, who live in silly little university towns sheltered from the fresh breezes of the wide open spaces). In this he is merely echoing Whitehead's 'literary superstition'.

The fact is that there are few more 'popular' subjects than mathematics. Most people have some appreciation of mathematics, just as most people can enjoy a pleasant tune; and there are probably more people really interested in mathematics than in music. Appearances may suggest the contrary, but there are easy explanations. Music can be used to stimulate mass emotion, while mathematics cannot; and musical incapacity is recognized (no doubt rightly) as mildly discreditable, whereas most people are so frightened of the name of mathematics that they are ready, quite unaffectedly, to exaggerate their own mathematical stupidity.

A very little reflection is enough to expose the absurdity of the 'literary superstition'. There are masses of chess-players in every civilized country — in Russia, almost the whole educated population; and every chess-player can recognize and appreciate a 'beautiful' game or problem. Yet a chess problem is *simply* an exercise in pure mathematics (a game not entirely, since psychology also plays a part), and everyone who calls a problem 'beautiful' is applauding mathematical beauty, even if it is beauty of a comparatively lowly kind. Chess problems are the hymn-tunes of mathematics.

We may learn the same lesson, at a lower level but for a wider public, from bridge, or descending further, from the puzzle columns of the popular newspapers. Nearly all their immense popularity is a tribute to the drawing power of rudimentary mathematics, and the better makers of puzzles, such as Dudeney[79] or 'Caliban',[80]

78 Hogben, 'Clarity is not enough', p. 107. Hardy misquotes 'exerts' as 'exercises'.
79 Henry Ernest Dudeney (1857–1930): mathematician and creator of logic puzzles.
80 The pen-name under which Hubert Phillips (1891–1964): economist and journalist, composed puzzles.

use very little else. They know their business; what the public wants is a little intellectual 'kick', and nothing else has quite the kick of mathematics.

I might add that there is nothing in the world which pleases even famous men (and men who have used disparaging language about mathematics) quite so much as to discover, or rediscover, a genuine mathematical theorem. Herbert Spencer[81] republished in his autobiography[82] a theorem about circles which he proved when he was twenty (not knowing that it had been proved over two thousand years before by Plato[83]).[84] Professor Soddy[85] is a more recent and a more striking example (but his theorem really is his own).[*]

11

A chess problem is genuine mathematics, but it is in some way 'trivial' mathematics. However ingenious and intricate, however original and surprising the moves, there is something

81 Herbert Spencer (1820–1903): philosopher, biologist, and political theorist.

82 Spencer, *An Autobiography*, vol. I, Appendix B; originally published as Spencer, 'Geometrical Theorem'.

83 Plato [Πλάτων *Plátōn*] (c. 429–c. 347 BCE): philosopher.

84 Mackay, 'Herbert Spencer and Mathematics' traces the theorem to the 18th century.

85 Frederick Soddy (1877–1956): radiochemist; 1921 Nobel laureate in chemistry. His 'Qui s'accuse s'acquitte' is a scathing review of *A Mathematician's Apology*; see pp. 106 sqq..

* See his letters on the 'Hexlet' in *Nature*, vols 137–9 (1936–7).[86]

86 For any three spheres A, B, and C that are mutually tangent, there exists a chain of six spheres $S_0, \dots, S_5$ such that each S_i is tangent to A, B, and C, and that each S_i is tangent to its neighbours S_{i-1} and S_{i+1} (taking subscripts modulo 6). Soddy's result states that for each i, the mean of the bends of S_i and S_{i+3} equals the sum of the bends of A, B, and C. (The bend of a sphere is the inverse of its radius.) Soddy first stated the result (in verse form) in 'The Hexlet (Dec. 1936)' and returned to it in 'The Hexlet (Jan. 1937)' and 'The Hexlet (Feb. 1937)'.

essential lacking. Chess problems are *unimportant*. The best mathematics is serious as well as beautiful — *important* if you like, but the word is very ambiguous, and 'serious' expresses what I mean much better.

I am not thinking of the 'practical' consequences of mathematics. I have to return to that point later: at present I will say only that if a chess problem is, in the crude sense, 'useless', then that is equally true of most of the best mathematics; that very little of mathematics is useful practically, and that that little is comparatively dull. The 'seriousness' of a mathematical theorem lies, not in its practical consequences, which are usually negligible, but in the *significance* of the mathematical ideas which it connects. We may say, roughly, that a mathematical idea is 'significant' if it can be connected, in a natural and illuminating way, with a large complex of other mathematical ideas. Thus a serious mathematical theorem, a theorem which connects significant ideas, is likely to lead to important advances in mathematics itself and even in other sciences. No chess problem has ever affected the general development of scientific thought; Pythagoras, Newton, Einstein have in their times changed its whole direction.

The seriousness of a theorem, of course, does not *lie in* its consequences, which are merely the *evidence* for its seriousness. Shakespeare[87] had an enormous influence on the development of the English language, Otway[88] next to none, but that is not why Shakespeare was the better poet. He was the better poet because he wrote much better poetry. The inferiority of the chess problem, like that of Otway's poetry, lies not in its consequences but in its content.

There is one more point which I shall dismiss very shortly, not because it is uninteresting but because it is difficult, and because I have no qualifications for any serious discussion in aesthetics. The beauty of a mathematical theorem *depends* a great deal on its seriousness, as even in poetry the beauty of a line may depend to some extent on the significance of the ideas which it contains. I

87 William Shakespeare (1564–1616): playwright and poet.
88 Thomas Otway (1652–85): playwright and poet.

quoted two lines of Shakespeare as an example of the sheer beauty of a verbal pattern; but

> 'After life's fitful fever he sleeps well'[89]

seems still more beautiful. The pattern is just as fine, and in this case the ideas have significance and the thesis is sound, so that our emotions are stirred much more deeply. The ideas do matter to the pattern, even in poetry, and much more, naturally, in mathematics; but I must not try to argue the question seriously.

12

It will be clear by now that, if we are to have any chance of making progress, I must produce examples of 'real' mathematical theorems, theorems which every mathematician will admit to be first-rate. And here I am very heavily handicapped by the restrictions under which I am writing. On the one hand my examples must be very simple, and intelligible to a reader who has no specialized mathematical knowledge; no elaborate preliminary explanations must be needed; and a reader must be able to follow the proofs as well as the enunciations. These conditions exclude, for instance, many of the most beautiful theorems of the theory of numbers, such as Fermat's[90] 'two square' theorem[91] or the law of quadratic reciprocity.[92] And on the other hand my examples should be drawn from 'pukka' mathematics, the mathematics of the working professional mathematician; and this condition excludes a

89 Shakespeare, *Macbeth*, Act 3, scene ii.

90 Pierre de Fermat (1607–65): jurist and amateur mathematician.

91 Hardy gives the statement of this result and discusses it further on p. 30.

92 The law of quadratic reciprocity states that if p and q are odd prime numbers, then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}},$$

good deal which it would be comparatively easy to make intelligible but which trespasses on logic and mathematical philosophy.

I can hardly do better than go back to the Greeks. I will state and prove two of the famous theorems of Greek mathematics. They are 'simple' theorems, simple both in idea and in execution, but there is no doubt at all about their being theorems of the highest class. Each is as fresh and significant as when it was discovered — two thousand years have not written a wrinkle on either of them. Finally, both the statements and the proofs can be mastered in an hour by any intelligent reader, however slender his mathematical equipment.

1. The first is Euclid's[*93] proof of the existence of an infinity of prime numbers.

The *prime numbers* or *primes* are the numbers

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots \qquad \text{(A)}$$

which cannot be resolved into smaller factors[†]. Thus 37 and 317 are prime. The primes are the material out of which all numbers are built up by multiplication: thus $666 = 2 \cdot 3 \cdot 3 \cdot 37$. Every number which is not prime itself is divisible by at least one prime (usually, of course, by several). We have to prove that there are infinitely many primes, i.e. that the series (A) never comes to an end.

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where, for any natural number a,

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \equiv x^2 \pmod{p} \text{ for some } x \not\equiv 0 \pmod{p}; \\ -1 & \text{if } a \not\equiv x^2 \pmod{p} \text{ for any } x; \\ 0 & \text{if } a \equiv 0 \pmod{p}. \end{cases}$$

Gauss, who first proved this result, said that 'it must be regarded as one of the most elegant of its type' (Gauss, *Disquisitiones Arithmeticae*, § 151). Gauss gave six different proofs, and many more have been found since; see Baumgart, *The Quadratic Reciprocity Law*.

* *Elements* IX 20. The real origin of many theorems in the *Elements* is obscure, but there seems to be no particular reason for supposing that this one is not Euclid's own.

93 Euclid [Εὐκλείδης *Eukleidēs*] (*fl.* 300 BCE): mathematician; author of the *Elements*.

† There are technical reasons for not counting 1 as a prime.

Let us suppose that it does, and that

$$2, 3, 5, \ldots, P$$

is the complete series (so that P is the largest prime); and let us, on this hypothesis, consider the number Q defined by the formula

$$Q = (2 \cdot 3 \cdot 5 \cdots P) + 1.$$

It is plain that Q is not divisible by any of $2, 3, 5, \ldots, P$; for it leaves the remainder 1 when divided by any one of these numbers. But, if not itself prime, it is divisible by *some* prime, and therefore there is a prime (which may be Q itself) greater than any of them. This contradicts our hypothesis, that there is no prime greater than P; and therefore this hypothesis is false.

The proof is by *reductio ad absurdum*, and *reductio ad absurdum*, which Euclid loved so much, is one of a mathematician's finest weapons[*]. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers *the game*.

* The proof can be arranged so as to avoid a *reductio*, and logicians of some schools would prefer that it should be.[94]

94 Hardy gives the theorem in the form 'There are infinitely many prime numbers' and begins his *reductio* by saying (essentially) 'Suppose there are only finitely many prime numbers'. Euclid's proposition is 'Prime numbers are more than any assigned multitude of prime numbers' and his proof proceeds to show that there is a prime number outside of any given multitude (or set) $p_1, p_2, \ldots, p_n$ of prime numbers. The *reductio* in Euclid's proof is confined to proving that a prime number that divides $\operatorname{lcm}(p_1 \cdot p_2 \cdots p_n) + 1$ cannot be any of the given prime numbers p_i: the *reductio* is 'local' to this step, not 'global' in the proof. Actually, Euclid's proof, as written, only considers a collection of three prime numbers, but it is clear that this collection stands for an arbitrary multitude of prime numbers. For the original proof with Euclid's notation, see Euclid, *Elements*, bk IX, Proposition 20. See Hardy & Woodgold, 'Prime simplicity' for a survey of just how common the *reductio* view is among mathematicians, logicians, and historians of mathematics. For a further study of how Euclid's proof has been changed in its modern presentation, see Siegmund-Schultze, 'Euclid's Proof'.

13

2. My second example is Pythagoras's[*] proof of the 'irrationality' of $\sqrt{2}$.

A 'rational number' is a fraction $\frac{a}{b}$, where a and b are integers; we may suppose that a and b have no common factor, since if they had we could remove it. To say that '$\sqrt{2}$ is irrational' is merely another way of saying that 2 cannot be expressed in the form $\left(\frac{a}{b}\right)^2$; and this is the same thing as saying that the equation

$$a^2 = 2b^2 \tag{B}$$

cannot be satisfied by integral values of a and b which have no common factor. This is a theorem of pure arithmetic, which does not demand any knowledge of 'irrational numbers' or depend on any theory about their nature.

We argue again by *reductio ad absurdum*; we suppose that (B) is true, a and b being integers without any common factor. It follows from (B) that a^2 is even (since $2b^2$ is divisible by 2), and therefore that a is even (since the square of an odd number is odd). If a is even then

$$a = 2c \tag{C}$$

for some integral value of c and therefore

$$2b^2 = a^2 = (2c)^2 = 4c^2$$

or

$$b^2 = 2c^2. \tag{D}$$

Hence b^2 is even, and therefore (for the same reason as before) b is even. That is to say, a and b are both even, and so have the

[*] The proof traditionally ascribed to Pythagoras, and certainly a product of his school. The theorem occurs, in a much more general form, in Euclid *Elements* x 9).

common factor 2. This contradicts our hypothesis, and therefore the hypothesis is false.

It follows from Pythagoras's theorem that the diagonal of a square is incommensurable with the side (that their ratio is not a rational number, that there is no unit of which both are integral multiples). For if we take the side as our unit of length, and the length of the diagonal is d, then, by a very familiar theorem also ascribed to Pythagoras*,

$$d^2 = 1^2 + 1^2 = 2,$$

so that d cannot be a rational number.

I could quote any number of fine theorems from the theory of numbers whose *meaning* anyone can understand. For example, there is what is called 'the fundamental theorem of arithmetic', that any integer can be resolved, in *one way only*, into a product of primes. Thus $666 = 2 \cdot 3 \cdot 3 \cdot 37$, and there is no other decomposition; it is impossible that $666 = 2 \cdot 11 \cdot 29$ or that $13 \cdot 89 = 17 \cdot 73$ (and we can see so without working out the products). This theorem is, as its name implies, the foundation of higher arithmetic; but the proof, although not 'difficult', requires a certain amount of preface and might be found tedious by an unmathematical reader.

Another famous and beautiful theorem is Fermat's 'two square' theorem. The primes may (if we ignore the special prime 2) be arranged in two classes; the primes

$$5, 13, 17, 29, 37, 41, \ldots$$

which leave remainder 1 when divided by 4, and the primes

$$3, 7, 11, 19, 23, 31, \ldots$$

which leave remainder 3. All the primes of the first class, and none of the second, can be expressed as the sum of two integral squares: thus

$$5 = 1^2 + 2^2, \qquad 13 = 2^2 + 3^2,$$
$$17 = 1^2 + 4^2, \qquad 29 = 2^2 + 5^2;$$

* Euclid, *Elements* I 47.

but 3, 7, 11, and 19 are not expressible in this way (as the reader may check by trial). This is Fermat's theorem, which is ranked, very justly, as one of the finest of arithmetic. Unfortunately there is no proof within the comprehension of anybody but a fairly expert mathematician.[95]

There are also beautiful theorems in the 'theory of aggregates' (*Mengenlehre*[96]), such as Cantor's[97] theorem of the 'non-enumerability' of the continuum. Here there is just the opposite difficulty. The proof is easy enough, when once the language has been mastered, but considerable explanation is necessary before the *meaning* of the theorem becomes clear. So I will not try to give more examples. Those which I have given are test cases, and a reader who cannot appreciate them is unlikely to appreciate anything in mathematics.

I said that a mathematician was a maker of patterns of ideas, and that beauty and seriousness were the criteria by which his patterns should be judged. I can hardly believe that anyone who has understood the two theorems will dispute that they pass these tests. If we compare them with Dudeney's most ingenious puzzles, or the finest chess problems that masters of that art have composed, their superiority in both respects stands out: there is an unmistakable difference of class. They are much more serious, and also much more beautiful; can we define, a little more closely, where their superiority lies?

95 There are many proofs of this result; see Dickson, *Diophantine Analysis*, pp. 227 sqq., or, for a comparison of three proofs, Avigad, 'Mathematical method and proof', § 2.3. An elementary proof was published by H. J. S. Smith in 1855 (Smith, 'De compositione numerorum primorum formae $4\lambda + 1$ ex duobus quadratis'), but Hardy was probably unaware of it (Clarke et al., 'H.J.S. Smith and the Fermat Two Squares Theorem', p. 652). There is now at least one proof that is accessible to to a reader who has no specialized mathematical knowledge: the proof in Zagier, 'A one-sentence proof' when written out in full, as in Aigner & Ziegler, *Proofs from THE BOOK* (1st edn), pp. 19–20.

96 Set theory.

97 Georg Ferdinand Ludwig Philipp Cantor (1845–1918): number theorist and founder of set theory.

14

In the first place, the superiority of the mathematical theorems in seriousness is obvious and overwhelming. The chess problem is the product of an ingenious but very limited complex of ideas, which do not differ from one another very fundamentally and have no external repercussions. We should think in the same way if chess had never been invented, whereas the theorems of Euclid and Pythagoras have influenced thought profoundly, even outside mathematics.

Thus Euclid's theorem is vital for the whole structure of arithmetic. The primes are the raw material out of which we have to build arithmetic, and Euclid's theorem assures us that we have plenty of material for the task. But the theorem of Pythagoras has wider applications and provides a better text. We should observe first that Pythagoras's argument is capable of far-reaching extension, and can be applied, with little change of principle, to very wide classes of 'irrationals'. We can prove very similarly (as Theodorus[98] seems to have done) that

$$\sqrt{3},\ \sqrt{5},\ \sqrt{7},\ \sqrt{11},\ \sqrt{13},\ \sqrt{17}$$

are irrational, or (going beyond Theodorus) that $\sqrt[3]{2}$ and $\sqrt[3]{17}$ are irrational[*].

Euclid's theorem tells us that we have a good supply of material for the construction of a coherent arithmetic of the integers. Pythagoras's theorem and its extensions tell us that, when we have

98 Theodorus of Cyrene [Θεόδωρος ὁ Κυρηναῖος] (*fl.* 5th century BCE): mathematician; appeared in three of Plato's dialogues: the *Theaetetus*, the *Sophist*, the *Politicus*. The source for the results that Hardy attributed to Theodorus is an account by Theaetetus in the first of these dialogues (*Theaetetus*, 147d). The account only explicitly mentions 3 and 5, and a linguistic ambiguity makes it unclear whether 17 was the last case he proved or the first he was unable to prove. For an extensive discussion, see Knorr, *The Evolution of the Euclidean Elements*, ch. III.

* See Ch. IV of Hardy and Wright's *Introduction to the Theory of Numbers*, where there are discussions of different generalizations of Pythagoras's argument, and of a historical puzzle about Theodorus.

constructed this arithmetic, it will not prove sufficient for our needs, since there will be many magnitudes which obtrude themselves upon our attention and which it will be unable to measure; the diagonal of the square is merely the most obvious example. The profound importance of this discovery was recognized at once by the Greek mathematicians. They had begun by assuming (in accordance, I suppose, with the 'natural' dictates of 'common sense') that all magnitudes of the same kind are commensurable, that any two lengths, for example, are multiples of some common unit, and they had constructed a theory of proportion based on this assumption. Pythagoras's discovery exposed the unsoundness of this foundation, and led to the construction of the much more profound theory of Eudoxus[99] which is set out in the fifth book of the *Elements*, and which is regarded by many modern mathematicians as the finest achievement of Greek mathematics. This theory is astonishingly modern in spirit, and may be regarded as the beginning of the modern theory of irrational number, which has revolutionized mathematical analysis and had much influence on recent philosophy.

There is no doubt at all, then, of the 'seriousness' of either theorem. It is therefore the better worth remarking that neither theorem has the slightest 'practical' importance. In practical applications we are concerned only with comparatively small numbers; only stellar astronomy and atomic physics deal with 'large' numbers, and they have very little more practical importance, as yet, than the most abstract pure mathematics. I do not know what is the highest degree of accuracy which is ever useful to an engineer — we shall be very generous if we say ten significant figures. Then

$$3.141\,592\,65$$

(the value of π to eight places of decimals) is the ratio

$$\frac{314\,159\,265}{100\,000\,000}$$

99 Eudoxus of Cnidus [Εὔδοξος *Eúdoxos*] (c. 390–c. 337 BCE): mathematician and astronomer.

of two numbers of nine digits. The number of primes less than 1 000 000 000 is 50 847 478: that is enough for an engineer, and he can be perfectly happy without the rest. So much for Euclid's theorem; and, as regards Pythagoras's, it is obvious that irrationals are uninteresting to an engineer, since he is concerned only with approximations, and all approximations are rational.

15

A 'serious' theorem is a theorem which contains 'significant' ideas, and I suppose that I ought to try to analyse a little more closely the qualities which make a mathematical idea significant. This is very difficult, and it is unlikely that any analysis which I can give will be very valuable. We can recognize a 'significant' idea when we see it, as we can those which occur in my two standard theorems; but this power of recognition requires a rather high degree of mathematical sophistication, and of that familiarity with mathematical ideas which comes only from many years spent in their company. So I must attempt some sort of analysis; and it should be possible to make one which, however inadequate, is sound and intelligible so far as it goes. There are two things at any rate which seem essential, a certain *generality* and a certain *depth*; but neither quality is easy to define at all precisely.

A significant mathematical idea, a serious mathematical theorem, should be 'general' in some such sense as this. The idea should be one which is a constituent in many mathematical constructs, which is used in the proof of theorems of many different kinds. The theorem should be one which, even if stated originally (like Pythagoras's theorem) in a quite special form, is capable of considerable extension and is typical of a whole class of theorems of its kind. The relations revealed by the proof should be such as connect many different mathematical ideas. All this is very vague, and subject to many reservations. But it is easy enough to see that a theorem is unlikely to be serious when it lacks these qualities conspicuously; we have only to take examples from the isolated curiosities in which arithmetic abounds. I take two, almost at random, from Rouse Ball's[100] *Mathematical Recreations*[*].

(*a*) 8712 and 9801 are the only four-figure numbers which are integral multiples of their 'reversals':

$$8712 = 4 \cdot 2178, \qquad 9801 = 9 \cdot 1089,$$

and there are no other numbers below 10 000 which have this property.

(*b*) There are just four numbers (after 1) which are the sums of the cubes of their digits, viz.

$$153 = 1^3 + 5^3 + 3^3, \qquad 370 = 3^3 + 7^3 + 0^3,$$
$$371 = 3^3 + 7^3 + 1^3, \qquad 407 = 4^3 + 0^3 + 7^3.$$

These are odd facts, very suitable for puzzle columns and likely to amuse amateurs, but there is nothing in them which appeals much to a mathematician. The proofs are neither difficult nor interesting — merely a little tiresome. The theorems are not serious; and it is plain that one reason (though perhaps not the most important) is the extreme speciality of both the enunciations and the proofs, which are not capable of any significant generalization.

16

'Generality' is an ambiguous and rather dangerous word, and we must be careful not to allow it to dominate our discussion too much. It is used in various senses both in mathematics and in writings about mathematics, and there is one of these in particular, on which logicians have very properly laid great stress,

100 Walter William Rouse Ball (1850–1925): mathematician, lawyer, and historian of mathematics.

* 11th edition, 1939 (revised by H. S. M. Coxeter[101]).

101 Harold Scott MacDonald Coxeter (1907–2003): geometer.

which is entirely irrelevant here. In this sense, which is quite easy to define, *all* mathematical theorems are equally and completely 'general'.

'The certainty of mathematics', says Whitehead[*], 'depends on its complete abstract generality.'[102] When we assert that 2 + 3 = 5, we are asserting a relation between three groups of 'things'; and these 'things' are not apples or pennies, or things of any one particular sort or another, but *just* things, 'any old things'. The meaning of the statement is entirely independent of the individualities of the members of the groups. All mathematical 'objects' or 'entities' or 'relations', such as '2', '3', '5', '+', or '=', and all mathematical propositions in which they occur, are completely general in the sense of being completely abstract. Indeed one of Whitehead's words is superfluous, since generality, in this sense, *is* abstractness.

This sense of the word is important, and the logicians are quite right to stress it, since it embodies a truism which a good many people who ought to know better are apt to forget. It is quite common, for example, for an astronomer or a physicist to claim that he has found a 'mathematical proof' that the physical universe must behave in a particular way. All such claims, if interpreted literally, are strictly nonsense. It *cannot* be possible to prove mathematically that there will be an eclipse to-morrow, because eclipses, and other physical phenomena, do not form part of the abstract world of mathematics; and this, I suppose, all astronomers would admit when pressed, however many eclipses they may have predicted correctly.

It is obvious that we are not concerned with this sort of 'generality' now. We are looking for differences of generality between one mathematical theorem and another, and in Whitehead's sense all are equally general. Thus the 'trivial' theorems (*a*) and (*b*) of § 16 are just as 'abstract' or 'general' as those of Euclid and Pythagoras, and so is a chess problem. It makes no difference to a chess problem whether the pieces are white and black, or red and green, or whether there are physical 'pieces' at all; it is the *same* problem

* Whitehead, *Science and the Modern World*, p. 33.

102 In the original, 'depends upon'.

which an expert carries easily in his head and which we have to reconstruct laboriously with the aid of the board. The board and the pieces are mere devices to stimulate our sluggish imaginations, and are no more essential to the problem than the blackboard and the chalk are to the theorems in a mathematical lecture.

It is not this kind of generality, common to all mathematical theorems, which we are looking for now, but the more subtle and elusive kind of generality which I tried to describe in rough terms in § 15. And we must be careful not to lay *too* much stress even on generality of this kind (as I think logicians like Whitehead tend to do). It is not mere 'piling of subtlety of generalization upon subtlety of generalization'[*] which is the outstanding achievement of modern mathematics. Some measure of generality must be present in any high-class theorem, but *too much* tends inevitably to insipidity. 'Everything is what it is, and not another thing',[103] and the differences between things are quite as interesting as their resemblances. We do not choose our friends because they embody all the pleasant qualities of humanity, but because they are the people that they are. And so in mathematics; a property common to too many objects can hardly be very exciting, and mathematical ideas also become dim unless they have plenty of individuality. Here at any rate I can quote Whitehead on my side: 'it is the large generalization, limited by a happy particularity, which is the fruitful conception.'[*][104]

17

The second quality which I demanded in a significant idea was *depth*, and this is still more difficult to define. It has *something* to do with *difficulty*; the 'deeper' ideas are usually the

* Whitehead, *Science and the Modern World*, p. 44.

103 Butler, *Fifteen Sermons*, p. xxvii. In the original, 'Every Thing is what it is, and not another Thing.'

* Whitehead, *Science and the Modern World*, p. 46.

104 This quotation is a complete sentence in the original.

harder to grasp: but it is not at all the same. The ideas underlying Pythagoras's theorem and its generalizations are quite deep, but no mathematician now would find them difficult. On the other hand a theorem may be essentially superficial and yet quite difficult to prove (as are many 'Diophantine'[105] theorems, i.e. theorems about the solution of equations in integers).

It seems that mathematical ideas are arranged somehow in strata, the ideas in each stratum being linked by a complex of relations both among themselves and with those above and below. The lower the stratum, the deeper (and in general the more difficult) the idea. Thus the idea of an 'irrational' is deeper than that of an integer; and Pythagoras's theorem is, for that reason, deeper than Euclid's. Let us concentrate our attention on the relations between the integers, or some other group of objects lying in some particular stratum. Then it may happen that one of these relations can be comprehended completely, that we can recognize and prove, for example, some property of the integers, without any knowledge of the contents of lower strata. Thus we proved Euclid's theorem by consideration of properties of integers only. But there are also many theorems about integers which we cannot appreciate properly, and still less prove, without digging deeper and considering what happens below.

It is easy to find examples in the theory of prime numbers. Euclid's theorem is very important, but not very deep: we can prove that there are infinitely many primes without using any notion deeper than that of 'divisibility'. But new questions suggest themselves as soon as we know the answer to this one. There is an infinity of primes, but how is this infinity distributed? Given a large number N, say 10^{80} or $10^{10^{10}}$,[*] about how many primes are there less than N?[†] When we ask *these* questions, we find ourselves in a quite

105 Diophantus of Alexandria [Διόφαντος] (c. 200–c. 285 CE): mathematician.

* It is supposed that the number of protons in the universe is about 10^{80}. The number $10^{10^{10}}$, if written at length, would occupy about 50 000 volumes of average size.

† As I mentioned in § 14, there are 50 847 478 primes less than 1 000 000 000; but that is as far as our *exact* knowledge extends.[106]

106 The number of primes less that 10^{26} is 1 699 246 750 872 437 141 327 603. At the

different position. We can answer them, with rather surprising accuracy, but only by boring much deeper, leaving the integers above us for a while, and using the most powerful weapons of the modern theory of functions. Thus the theorem which answers our questions (the so-called 'Prime Number Theorem') is a much deeper theorem than Euclid's or even Pythagoras's.

I could multiply examples, but this notion of 'depth' is an elusive one even for a mathematician who can recognize it, and I can hardly suppose that I could say anything more about it here which would be of much help to other readers.

18

There is still one point remaining over from § 11, where I started the comparison between 'real mathematics' and chess. We may take it for granted now that in substance, seriousness, significance, the advantage of the real mathematical theorem is overwhelming. It is almost equally obvious, to a trained intelligence, that it has a great advantage in beauty also; but this advantage is much harder to define or locate, since the *main* defect of the chess problem is plainly its 'triviality', and the contrast in this respect mingles with and disturbs any more purely aesthetic judgement. What 'purely aesthetic' qualities can we distinguish in such theorems as Euclid's and Pythagoras's? I will not risk more than a few disjointed remarks.

In both theorems (and in the theorems, of course, I include the proofs) there is a very high degree of *unexpectedness*, combined with *inevitability* and *economy*. The arguments take so odd and surprising a form; the weapons used seem so childishly simple when compared with the far-reaching results; but there is no escape from the conclusions. There are no complications of detail — one line of attack is enough in each case; and this is true too of the

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time of annotation, 10^{26} is the highest power of 10 for which the number of primes less than that power is known.

proofs of many much more difficult theorems, the full appreciation of which demands quite a high degree of technical proficiency. We do not want many 'variations' in the proof of a mathematical theorem: 'enumeration of cases', indeed, is one of the duller forms of mathematical argument. A mathematical proof should resemble a simple and clear-cut constellation, not a scattered cluster in the Milky Way.

A chess problem also has unexpectedness, and a certain economy; it is essential that the moves should be surprising, and that every piece on the board should play its part. But the aesthetic effect is cumulative. It is essential also (unless the problem is too simple to be really amusing) that the key-move should be followed by a good many variations, each requiring its own individual answer. 'If P–B5 then Kt–R6; if … then … ; if … then … .' — the effect would be spoilt if there were not a good many different replies. All this is quite genuine mathematics, and has its merits; but it is just that 'proof by enumeration of cases' (and of cases which do not, at bottom, differ at all profoundly*) which a real mathematician tends to despise. I am inclined to think that I could reinforce my argument by appealing to the feelings of chess-players themselves. Surely a chess master, a player of great games and great matches, at bottom scorns a problemist's purely mathematical art. He has much of it in reserve himself, and can produce it in an emergency: 'if he had made such and such a move, then I had such and such a winning combination in mind.' But the 'great game' of chess is primarily psychological, a conflict between one trained intelligence and another, and not a mere collection of small mathematical theorems.

19

I must return to my Oxford apology, and examine a little more carefully some of the points which I postponed in § 6. It

* I believe that it is now regarded as a merit in a problem that there should be many variations of the same type.

will be obvious by now that I am interested in mathematics only as a creative art. But there are other questions to be considered, and in particular that of the 'utility' (or uselessness) of mathematics, about which there is much confusion of thought. We must also consider whether mathematics is really quite so 'harmless' as I took for granted in my Oxford lecture.

A science or an art may be said to be 'useful' if its development increases, even indirectly, the material well-being and comfort of men, if it promotes happiness, using that word in a crude and commonplace way. Thus medicine and physiology are useful because they relieve suffering, and engineering is useful because it helps us to build houses and bridges, and so to raise the standard of life (engineering, of course, does harm as well, but that is not the question at the moment). Now some mathematics is certainly useful in this way; the engineers could not do their job without a fair working knowledge of mathematics, and mathematics is beginning to find applications even in physiology. So here we have a possible ground for a defence of mathematics; it may not be the best, or even a particularly strong defence, but it is one which we must examine. The 'nobler' uses of mathematics, if such they be, the uses which it shares with all creative art, will be irrelevant to our examination. Mathematics may, like poetry or music, 'promote and sustain a lofty habit of mind',[107] and so increase the happiness of mathematicians and even of other people; but to defend it on that ground would be merely to elaborate what I have said already. What we have to consider now is the 'crude' utility of mathematics.

20

All this may seem very obvious, but even here there is often a good deal of confusion, since the most 'useful' subjects are

107 Hardy is probably thinking here of a phrase of Russell's: 'Every great study is [...] a means of creating and sustaining a lofty habit of mind' (Russell, 'The Study of Mathematics', p. 73).

quite commonly just those which it is most useless for most of us to learn. It is useful to have an adequate supply of physiologists and engineers; but physiology and engineering are not useful studies for ordinary men (though their study may of course be defended on other grounds). For my own part I have never once found myself in a position where such scientific knowledge as I possess, outside pure mathematics, has brought me the slightest advantage.

It is indeed rather astonishing how little practical value scientific knowledge has for ordinary men, how dull and commonplace such of it as has value is, and how its value seems almost to vary inversely to its reputed utility. It is useful to be tolerably quick at common arithmetic (and that, of course, is pure mathematics). It is useful to know a little French or German, a little history and geography, perhaps even a little economics. But a little chemistry, physics, or physiology has no value at all in ordinary life. We know that the gas will burn without knowing its constitution; when our cars break down we take them to a garage; when our stomach is out of order, we go to a doctor or a drugstore. We live either by rule of thumb or on other people's professional knowledge.

However, this is a side issue, a matter of pedagogy, interesting only to schoolmasters who have to advise parents clamouring for a 'useful' education for their sons. Of course we do not mean, when we say that physiology is useful, that most people ought to study physiology, but that the development of physiology by a handful of experts will increase the comfort of the majority. The questions which are important for us now are, how far mathematics can claim this sort of utility, what kinds of mathematics can make the strongest claims, and how far the intensive study of mathematics, as it is understood by mathematicians, can be justified on this ground alone.

21

It will probably be plain by now to what conclusions I am coming; so I will state them at once dogmatically and then elaborate them a little. It is undeniable that a good deal of elementary mathematics — and I use the word 'elementary' in the sense in which professional mathematicians use it, in which it includes, for example, a fair working knowledge of the differential and integral calculus — has considerable practical utility. These parts of mathematics are, on the whole, rather dull; they are just the parts which have least aesthetic value. The 'real' mathematics of the 'real' mathematicians, the mathematics of Fermat and Euler[108] and Gauss and Abel and Riemann, is almost wholly 'useless' (and this is as true of 'applied' as of 'pure' mathematics). It is not possible to justify the life of any genuine professional mathematician on the ground of the 'utility' of his work.

But here I must deal with a misconception. It is sometimes suggested that pure mathematicians glory in the uselessness of their work[*], and make it a boast that it has no practical applications. The

108 Leonhard Euler (1707–83): mathematician and physicist.

* I have been accused of taking this view myself. I once said that 'a science is said to be useful if its development tends to accentuate the existing inequalities in the distribution of wealth, or more directly promotes the destruction of human life',[109] and this sentence, written in 1915, has been quoted (for or against me) several times. It was of course a conscious rhetorical flourish, though one perhaps excusable at the time when it was written.

109 Hardy, 'Prime Numbers', p. 350. This is from the opening of Hardy's lecture to the 1915 meeting of the British Association for the Advancement of Science, and its sustained irony deserves quotation in full:

> 'The Theory of Numbers has always been regarded as one of the most obviously useless branches of Pure Mathematics. The accusation is one against which there is no valid defence; and it is never more just than when directed against the parts of the theory which are more particularly concerned with primes. A science is said to be useful if its development tends to accentuate the existing inequalities in the distribution of wealth, or more directly promotes the destruction of human life. The theory of prime numbers satisfies no such criteria. Those who pursue it will, if they are wise, make no attempt to justify their interest in a subject so trivial and so remote, and will console themselves with the thought that the greatest mathematicians of all

imputation is usually based on an incautious saying attributed to Gauss, to the effect that, if mathematics is the queen of the sciences, then the theory of numbers is, because of its supreme uselessness, the queen of mathematics — I have never been able to find an exact quotation.[110] I am sure that Gauss's saying (if indeed it be his) has been rather crudely misinterpreted. If the theory of numbers could be employed for any practical and obviously honourable purpose, if it could be turned directly to the furtherance of human happiness or the relief of human suffering, as physiology and even chemistry can, then surely neither Gauss nor any other mathematician would have been so foolish as to decry or regret such applications. But science works for evil as well as for good (and particularly, of course, in time of war); and both Gauss and lesser mathematicians may be justified in rejoicing that there is one science at any rate, and that their own, whose very remoteness from ordinary human activities should keep it gentle and clean.

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ages have found in it a mysterious attraction impossible to resist.'

110 The saying appears to originate in the biography of Gauss by Sartorius:

> 'Die Mathematik hielt Gauss um seine eigenen Worte zu gebrauchen für die Königin der Wissenschaften und die Arithmetik für die Königin der Mathematik. Diese lasse sich dann öfter herab der Astronomie und andern Naturwissenschaften einen Dienst zu erweisen, doch gebführe ihr unter allen Verhältnissen dar erste Rang.' (Sartorius von Waltershausen, *Gauss zum Gedächtniss*, p. 79)

In the English translation by Helen Worthington Gauss, great-granddaughter of Carl Friedrich, this is:

> 'To use Gauss' own words, mathematics was for him "the Queen of sciences, and arithmetic the Queen of mathematics". It may often stoop to do a service for astronomy and other natural sciences, but under all circumstances it must take first place.' (Sartorius von Waltershausen, *Carl Friedrich Gauss*, pp. 64–5)

There is no mention of 'uselessness'.

22

There is another misconception against which we must guard. It is quite natural to suppose that there is a great difference in utility between 'pure' and 'applied' mathematics. This is a delusion: there is a sharp distinction between the two kinds of mathematics, which I will explain in a moment, but it hardly affects their utility.

How do pure and applied mathematics differ from one another? This is a question which can be answered definitely and about which there is general agreement among mathematicians. There will be nothing in the least unorthodox about my answer, but it needs a little preface.

My next two sections will have a mildly philosophical flavour. The philosophy will not cut deep, or be in any way vital to my main theses; but I shall use words which are used very frequently with definite philosophical implications, and a reader might well become confused if I did not explain how I shall use them.

I have often used the adjective 'real', and as we use it commonly in conversation. I have spoken of 'real mathematics' and 'real mathematicians', as I might have spoken of 'real poetry' or 'real poets', and I shall continue to do so. But I shall also use the word 'reality', and with two different connotations.

In the first place, I shall speak of 'physical reality', and here again I shall be using the word in the ordinary sense. By physical reality I mean the material world, the world of day and night, earthquakes and eclipses, the world which physical science tries to describe.

I hardly suppose that, up to this point, any reader is likely to find trouble with my language, but now I am near to more difficult ground. For me, and I suppose for most mathematicians, there is another reality, which I will call 'mathematical reality'; and there is no sort of agreement about the nature of mathematical reality among either mathematicians or philosophers. Some hold that it is 'mental' and that in some sense we construct it, others that it is outside and independent of us. A man who could give a convincing account of mathematical reality would have solved very many of the most difficult problems of metaphysics. If he could include physical reality in his account, he would have solved them all.

I should not wish to argue any of these questions here even if I were competent to do so, but I will state my own position dogmatically in order to avoid minor misapprehensions. I believe that mathematical reality lies outside us,[111] that our function is to discover or *observe* it,[112] and that the theorems which we prove, and which we describe grandiloquently as our 'creations', are simply our notes of our observations. This view has been held, in one form or another, by many philosophers of high reputation from Plato onwards, and I shall use the language which is natural to a man who holds it. A reader who does not like the philosophy can alter the language: it will make very little difference to my conclusions.

23

The contrast between pure and applied mathematics stands out most clearly, perhaps, in geometry. There is the science of pure geometry[*], in which there are many geometries, projective geometry, Euclidean geometry, non-Euclidean geometry, and so forth. Each of these geometries is a *model*, a pattern of ideas, and is to be judged by the interest and beauty of its particular pattern. It is a *map* or *picture*, the joint product of many hands, a partial and imperfect copy (yet exact so far as it extends) of a section of mathematical reality. But the point which is important to us now is this, that there is one thing at any rate of which pure geometries are not pictures, and that is the spatio-temporal reality of the physical world. It is obvious, surely, that they cannot be, since earthquakes and eclipses are not mathematical concepts.

111 Hardy held that any philosophy acceptable to a mathematician must admit, in some way, that mathematics is part of objective reality (Hardy, 'Mathematical proof', p. 4).

112 For Hardy, proofs are ultimately just psychological devices to aid the observer; see *ibid.*, p. 18.

* We must of course, for the purposes of this discussion, count as pure geometry what mathematicians call 'analytical' geometry.

This may sound a little paradoxical to an outsider, but it is a truism to a geometer; and I may perhaps be able to make it clearer by an illustration. Let us suppose that I am giving a lecture on some system of geometry, such as ordinary Euclidean geometry, and that I draw figures on the blackboard to stimulate the imagination of my audience, rough drawings of straight lines or circles or ellipses. It is plain, first, that the truth of the theorems which I prove is in no way affected by the quality of my drawings. Their function is merely to bring home my meaning to my hearers, and, if I can do that, there would be no gain in having them redrawn by the most skilful draughtsman. They are pedagogical illustrations, not part of the real subject-matter of the lecture.

Now let us go a stage further. The room in which I am lecturing is part of the physical world, and has itself a certain pattern. The study of that pattern, and of the general pattern of physical reality, is a science in itself, which we may call 'physical geometry'. Suppose now that a violent dynamo, or a massive gravitating body, is introduced into the room. Then the physicists tell us that the geometry of the room is changed, its whole physical pattern slightly but definitely distorted. Do the theorems which I have proved become false? Surely it would be nonsense to suppose that the proofs of them which I have given are affected in any way. It would be like supposing that a play of Shakespeare is changed when a reader spills his tea over a page. The play is independent of the pages on which it is printed, and 'pure geometries' are independent of lecture rooms, or of any other detail of the physical world.

This is the point of view of a pure mathematician. Applied mathematicians, mathematical physicists, naturally take a different view, since they are preoccupied with the physical world itself, which also has its structure or pattern. We cannot describe this pattern exactly, as we can that of a pure geometry, but we can say something significant about it. We can describe, sometimes fairly accurately, sometimes very roughly, the relations which hold between some of its constituents, and compare them with the exact relations holding between constituents of some system of pure geometry. We may be able to trace a certain resemblance between the two sets of relations, and then the pure geometry will become interesting to physicists; it will give us, to that extent, a map which 'fits the facts' of the physical world. The geometer offers to the physicist a whole set of maps from which to choose. One map, perhaps, will fit the facts better than others, and then the geometry which provides that particular map will be the geometry most important for applied mathematics. I may add that even a pure mathematician may find his appreciation of this geometry quickened, since there is no mathematician so pure that he feels no interest at all in the physical world; but, in so far as he succumbs to this temptation, he will be abandoning his purely mathematical position.

24

There is another remark which suggests itself here and which physicists may find paradoxical, though the paradox will probably seem a good deal less than it did eighteen years ago.[113] I will express it in much the same words which I used in 1922 in an address[114] to Section A of the British Association.[115] My audience then was composed almost entirely of physicists, and I may have spoken a little provocatively on that account; but I would still stand by the substance of what I said.

I began by saying that there is probably less difference between the positions of a mathematician and of a physicist than is generally supposed, and that the most important seems to me to be this, that the mathematician is in much more direct contact with reality. This may seem a paradox, since it is the physicist who deals with the subject-matter usually described as 'real'; but a very little reflection is enough to show that the physicist's reality, whatever it may be,

113 This presumably refers to the emergence during the 1920s and 1930s of quantum mechanics and in particular notions such as Heisenberg's uncertainty principle.

114 See Hardy, 'The theory of numbers'.

115 That is, the British Association for the Advancement of Science. Section A is 'Mathematics and Physics'.

has few or none of the attributes which common sense ascribes instinctively to reality. A chair may be a collection of whirling electrons, or an idea in the mind of God: each of these accounts of it may have its merits, but neither conforms at all closely to the suggestions of common sense.

I went on to say that neither physicists nor philosophers have ever given any convincing account of what 'physical reality' is, or of how the physicist passes, from the confused mass of fact or sensation with which he starts, to the construction of the objects which he calls 'real'. Thus we cannot be said to know what the subject-matter of physics is; but this need not prevent us from understanding roughly what a physicist is trying to do. It is plain that he is trying to correlate the incoherent body of crude fact confronting him with some definite and orderly scheme of abstract relations, the kind of scheme which he can borrow only from mathematics.

A mathematician, on the other hand, is working with his own mathematical reality. Of this reality, as I explained in § 22, I take a 'realistic' and not an 'idealistic' view. At any rate (and this was my main point) this realistic view is much more plausible of mathematical than of physical reality, because mathematical objects are so much more what they seem. A chair or a star is not in the least like what it seems to be; the more we think of it, the fuzzier its outlines become in the haze of sensation which surrounds it; but '2' or '317' has nothing to do with sensation, and its properties stand out the more clearly the more closely we scrutinize it. It may be that modern physics fits best into some framework of idealistic philosophy — I do not believe it, but there are eminent physicists who say so. Pure mathematics, on the other hand, seems to me a rock on which all idealism founders: 317 is a prime, not because we think so, or because our minds are shaped in one way rather than another, but *because it is so*, because mathematical reality is built that way.

25

These distinctions between pure and applied mathematics are important in themselves, but they have very little bearing on our discussion of the 'usefulness' of mathematics. I spoke in § 21 of the 'real' mathematics of Fermat and other great mathematicians, the mathematics which has permanent aesthetic value, as for example the best Greek mathematics has, the mathematics which is eternal because the best of it may, like the best literature, continue to cause intense emotional satisfaction to thousands of people after thousands of years. These men were all primarily pure mathematicians (though the distinction was naturally a good deal less sharp in their days than it is now); but I was not thinking only of pure mathematics. I count Maxwell[116] and Einstein, Eddington[117] and Dirac,[118] among 'real' mathematicians.[119] The great modern achievements of applied mathematics have been in relativity and quantum mechanics, and these subjects are, at present at any rate, almost as 'useless' as the theory of numbers. It is the dull and elementary parts of applied mathematics, as it is the dull and elementary parts of pure mathematics, that work for good or ill. Time may change all this.[120] No one foresaw the applications of matrices and groups and other purely mathematical theories to modern physics, and it may be that some of the 'highbrow' applied

116 James Clerk Maxwell (1831–1879): physicist and mathematician.

117 Arthur Stanley Eddington (1882–1944): astronomer and physicist. Eddington reviewed the *Apology*; see p. 102.

118 Paul Adrien Maurice Dirac (1902–84): physicist; made important contributions to the development of quantum theory.

119 According to his friend C. P. Snow, Hardy admired many theoretical physicists 'beyond measure' and 'had a veneration for Einstein [...]; he thought him probably the greatest human being he'd ever met' (Snow, 'The Classical Mind', p. 813), but that among them only Dirac could have become 'a really good pure mathematician' (*loc. cit.*).

120 Note that this single sentence makes clear that, for Hardy, the link between aesthetic value and uselessness was *contingent*, not necessary: beautiful mathematics *happened to be* (in his view) useless. This point has been frequently misunderstood; see the discussion in the annotator's essay 'Legacy of the *Apology*', pp. 110 sqq..

mathematics will become 'useful' in as unexpected a way; but the evidence so far points to the conclusion that, in one subject as in the other, it is what is commonplace and dull that counts for practical life.

I can remember Eddington giving a happy example of the unattractiveness of 'useful' science. The British Association held a meeting in Leeds, and it was thought that the members might like to hear something of the applications of science to the 'heavy woollen' industry.[121] But the lectures and demonstrations arranged for this purpose were rather a fiasco. It appeared that the members (whether citizens of Leeds or not) wanted to be entertained, and that 'heavy wool' is not at all an entertaining subject. So the attendance at these lectures was very disappointing; but those who lectured on the excavations at Knossos, or on relativity, or on the theory of prime numbers, were delighted by the audiences that they drew.[122]

26

What parts of mathematics are useful?

First, the bulk of school mathematics, arithmetic, elementary algebra, elementary Euclidean geometry, elementary differential and integral calculus. We must except a certain amount of what is taught to 'specialists', such as projective geometry. In applied mathematics, the elements of mechanics (electricity, as taught in schools, must be classified as physics).

121 Leeds was a major centre for textile production.

122 This must refer to the 1927 meeting of the British Association for the Advancement of Science, for the previous meeting in Leeds was in 1890, when Eddington was still a child. However, another account states that 'the special sessions for the discussion of the science and technology of textile fabrics at the British Association was undoubtedly a success. The meetings were well attended, and attracted a number of scientific workers not engaged in textile research.' ('The British Association Meetings at Leeds'). Furthermore, the proceedings (*British Association 1927 Report*) seem not to mention lectures on Knossos.

Next, a fair proportion of university mathematics is also useful, that part of it which is really a development of school mathematics with a more finished technique, and a certain amount of the more physical subjects such as electricity and hydromechanics. We must also remember that a reserve of knowledge is always an advantage, and that the most practical of mathematicians may be seriously handicapped if his knowledge is the bare minimum which is essential to him; and for this reason we must add a little under every heading. But our general conclusion must be that such mathematics is useful as is wanted by a superior engineer or a moderate physicist; and that is roughly the same thing as to say, such mathematics as has no particular aesthetic merit. Euclidean geometry, for example, is useful in so far as it is dull — we do not want the axiomatics of parallels, or the theory of proportion, or the construction of the regular pentagon.

One rather curious conclusion emerges, that pure mathematics is on the whole distinctly more useful than applied. A pure mathematician seems to have the advantage on the practical as well as on the aesthetic side. For what is useful above all is *technique*, and mathematical technique is taught mainly through pure mathematics.

I hope that I need not say that I am not trying to decry mathematical physics, a splendid subject with tremendous problems where the finest imaginations have run riot. But is not the position of an ordinary applied mathematician in some ways a little pathetic? If he wants to be useful, he must work in a humdrum way, and he cannot give full play to his fancy even when he wishes to rise to the heights. 'Imaginary' universes are so much more beautiful than this stupidly constructed 'real' one; and most of the finest products of an applied mathematician's fancy must be rejected, as soon as they have been created, for the brutal but sufficient reason that they do not fit the facts.

The general conclusion, surely, stands out plainly enough. If useful knowledge is, as we agreed provisionally to say, knowledge which is likely, now or in the comparatively near future, to contribute to the material comfort of mankind, so that mere intellectual satisfaction is irrelevant, then the great bulk of higher mathematics is useless. Modern geometry and algebra, the theory of numbers, the theory of aggregates and functions, relativity, quantum mechanics — no one of them stands the test much better than another, and there is no real mathematician whose life can be justified on this ground. If this be the test, then Abel, Riemann, and Poincaré[123] wasted their lives; their contribution to human comfort was negligible, and the world would have been as happy a place without them.

27

It may be objected that my concept of 'utility' has been too narrow, that I have defined it in terms of 'happiness' or 'comfort' only, and have ignored the general 'social' effects of mathematics on which recent writers, with very different sympathies, have laid so much stress. Thus Whitehead (who has been a mathematician) speaks of 'the tremendous effect of mathematical knowledge on the lives of men, on their daily avocations, on the organization of society';[124] and Hogben (who is as unsympathetic to what I and other mathematicians call mathematics as Whitehead is sympathetic) says that 'without a knowledge of mathematics, the grammar of size and order, we cannot plan the rational society in which there will be leisure for all and poverty for none'[125] (and much more to the same effect).

I cannot really believe that all this eloquence will do much to comfort mathematicians. The language of both writers is violently exaggerated, and both of them ignore very obvious distinctions.

123 Jules Henri Poincaré (1854–1912): mathematician, physicist, and philosopher of science.

124 Whitehead, *Science and the Modern World*, p. 21. Whitehead actually speaks of 'the tremendous future effect'.

125 Hogben, *Mathematics for the Million*, p. 20. The quotation is a complete sentence. This passage was removed in the fourth edition.

This is very natural in Hogben's case, since he is admittedly not a mathematician; he means by 'mathematics' the mathematics which he can understand, and which I have called 'school' mathematics. This mathematics has many uses, which I have admitted, which we can call 'social' if we please, and which Hogben enforces with many interesting appeals to the history of mathematical discovery. It is this which gives his book its merit, since it enables him to make plain, to many readers who never have been and never will be mathematicians, that there is more in mathematics than they thought. But he has hardly any understanding of 'real' mathematics (as any one who reads what he says about Pythagoras's theorem, or about Euclid and Einstein, can tell at once[126]), and still less sympathy with it (as he spares no pains to show). 'Real' mathematics is to him merely an object of contemptuous pity.

It is not lack of understanding or of sympathy which is the trouble in Whitehead's case; but he forgets, in his enthusiasm, distinctions with which he is quite familiar. The mathematics which has this 'tremendous effect' on the 'daily avocations of men' and on 'the organization of society' is not the Whitehead but the Hogben mathematics. The mathematics which can be used 'for ordinary purposes by ordinary men' is negligible, and that which can be used by economists or sociologists hardly rises to 'scholarship standard'. The Whitehead mathematics may affect astronomy or physics profoundly, philosophy very appreciably — high thinking of one kind is always likely to affect high thinking of another — but it has extremely little effect on anything else. Its 'tremendous effects' have been, not on men generally, but on men like Whitehead himself.

126 Hogben's account of the irrationality of $\sqrt{2}$ is embedded in a discussion of the practicalities of finding approximations to square roots (Hogben, *Mathematics for the Million*, p. 94). His discussion of Euclid is likewise concerned with applications: 'We now know that the geometry of Euclid does not give us the best possible way of measuring space. This does not mean that it is not a useful branch of knowledge. It was and still is. New discoveries have simply taught us that it has its limitations' (*ibid.*, p. 114). Further, he described Euclid's view of space as a theory that 'has been brought down to earth by Einstein' (*ibid.*, p. 27).

28

There are then two mathematics. There is the real mathematics of the real mathematicians, and there is what I will call the 'trivial' mathematics, for want of a better word. The trivial mathematics may be justified by arguments which would appeal to Hogben, or other writers of his school, but there is no such defence for the real mathematics, which must be justified as art if it can be justified at all. There is nothing in the least paradoxical or unusual in this view, which is that held commonly by mathematicians.

We have still one more question to consider. We have concluded that the trivial mathematics is, on the whole, useful, and that the real mathematics, on the whole, is not; that the trivial mathematics does, and the real mathematics does not, 'do good' in a certain sense; but we have still to ask whether either sort of mathematics does *harm*. It would be paradoxical to suggest that mathematics of any sort does much harm in time of peace, so that we are driven to the consideration of the effects of mathematics on war. It is very difficult to argue such questions at all dispassionately now, and I should have preferred to avoid them; but some sort of discussion seems inevitable. Fortunately, it need not be a long one.

There is one comforting conclusion which is easy for a real mathematician. Real mathematics has no effects on war. No one has yet discovered any warlike purpose to be served by the theory of numbers or relativity, and it seems very unlikely that anyone will do so for many years.[127] It is true that there are branches of

127 It has become almost a cliché to note, in response to Hardy, that number theory has found applications in areas such as cryptography and the kind of theoretical physics exemplified by relativity and quantum mechanics ultimately led to the development of atomic weapons. Even the Rogers–Ramanujan identities, about which Hardy said 'it would be difficult to find more beautiful formulæ' (Hardy, 'Srinivasa Ramanujan', p. xxxiv; see also the discussion in the annotator's essay 'Context of the Apology', p. 93), have found application in physics (Baxter, 'Rogers–Ramanujan Identities'). However, Hardy held that there was only a contingent connection between real mathematics ('the mathematics which has permanent aesthetic value') and uselessness (see § 25 and n. 120); Hardy was thus only wrong about the probability of and time-scale for the emergence of applications in war. For a detailed discussion of this point, see the annotator's

applied mathematics, such as ballistics and aerodynamics, which have been developed deliberately for war and demand a quite elaborate technique: it is perhaps hard to call them 'trivial', but none of them has any claim to rank as 'real'. They are indeed repulsively ugly and intolerably dull; even Littlewood could not make ballistics respectable,[128] and if he could not who can?[129] So a real mathematician has his conscience clear; there is nothing to be set against any value his work may have; mathematics is, as I said at Oxford, a 'harmless and innocent'[130] occupation.

The trivial mathematics, on the other hand, has many applications in war. The gunnery experts and aeroplane designers, for example, could not do their work without it. And the general effect of these applications is plain: mathematics facilitates (if not so obviously as physics or chemistry) modern, scientific, 'total' war.

It is not so clear as it might seem that this is to be regretted, since there are two sharply contrasted views about modern scientific war. The first and the most obvious is that the effect of science on war is merely to magnify its horror, both by increasing the sufferings of the minority who have to fight and by extending them to other classes. This is the most natural and the orthodox view. But there is a very different view which seems also quite tenable, and which has been stated with great force by Haldane[131] in *Callinicus*.[*] It can be maintained that modern warfare is *less* horrible than the warfare of pre-scientific times; that bombs are probably more merciful than

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essay 'Legacy of the *Apology*', pp. 110 sqq..

128 Hardy's collaborator Littlewood worked on computing ballistics range tables during the First World War; see Littlewood, 'Adventures in Ballistics, 1915–1918. I'.

129 Hardy's student and obituarist Titchmarsh thought that Hardy's view of branches of applied mathematics developed for war, such as ballistics and aerodynamics, was influenced by his hatred of war (Titchmarsh, 'Godfrey Harold Hardy', p. 85). Newman interpreted this as meaning that Hardy's hatred of war was a *cause* of his evalution of these fields (J. R. Newman, *The World of Mathematics*, vol. 4, p. 2025).

130 See § 6.

131 John Burdon Sanderson Haldane (1892–1964): geneticist, physiologist, and mathematician.

* J. B. S. Haldane, *Callinicus*: *a Defence of Chemical Warfare* (1924).

bayonets; that lachrymatory gas and mustard gas are perhaps the most humane weapons yet devised by military science; and that the orthodox view rests solely on loose-thinking sentimentalism*. It may also be urged (though this was not one of Haldane's theses) that the equalization of risks which science was expected to bring would be in the long run salutary; that a civilian's life is not worth more than a soldier's, nor a woman's than a man's; that anything is better than the concentration of savagery on one particular class; and that, in short, the sooner war comes 'all out' the better.

I do not know which of these views is nearer to the truth. It is an urgent and a moving question, but I need not argue it here. It concerns only the 'trivial' mathematics, which it would be Hogben's business to defend rather than mine. The case for his mathematics may be rather more than a little soiled; the case for mine is unaffected.

Indeed, there is more to be said, since there is one purpose at any rate which the real mathematics may serve in war. When the world is mad, a mathematician may find in mathematics an incomparable anodyne. For mathematics is, of all the arts and sciences, the most austere and the most remote, and a mathematician should be of all men the one who can most easily take refuge where, as Bertrand Russell says, 'one at least of our nobler impulses can best escape from the dreary exile of the actual world.'[132] It is a pity that it should be necessary to make one very serious reservation — he must not be too old. Mathematics is not a contemplative but a creative subject; no one can draw much consolation from it when he has lost the power or the desire to create; and that is apt to happen to a mathematician rather soon. It is a pity, but in that case he does not matter a great deal anyhow, and it would be silly to bother about him.

* I do not wish to prejudge the question by this much misused word; it may be used quite legitimately to indicate certain types of unbalanced emotion. Many people, of course, use 'sentimentalism' as a term of abuse for other people's decent feelings, and 'realism' as a disguise for their own brutality.

132 Russell, 'The Study of Mathematics', p. 61.

29

I will end with a summary of my conclusions, but putting them in a more personal way. I said at the beginning that anyone who defends his subject will find that he is defending himself; and my justification of the life of a professional mathematician is bound to be, at bottom, a justification of my own. Thus this concluding section will be in its substance a fragment of autobiography.

I cannot remember ever having wanted to be anything but a mathematician. I suppose that it was always clear that my specific abilities lay that way, and it never occurred to me to question the verdict of my elders. I do not remember having felt, as a boy, any *passion* for mathematics, and such notions as I may have had of the career of a mathematician were far from noble. I thought of mathematics in terms of examinations and scholarships: I wanted to beat other boys, and this seemed to be the way in which I could do so most decisively.

I was about fifteen when (in a rather odd way) my ambitions took a sharper turn. There is a book by 'Alan St Aubyn'[*] called *A Fellow of Trinity*, one of a series dealing with what is supposed to be Cambridge college life. I suppose that it is a worse book than most of Marie Corelli's;[134] but a book can hardly be entirely bad if it fires a clever boy's imagination. There are two heroes, a primary hero called Flowers, who is almost wholly good, and a secondary hero, a much weaker vessel, called Brown. Flowers and Brown find many dangers in university life, but the worst is a gambling saloon in Chesterton[†][135] run by the Misses Bellenden, two fascinating but extremely wicked young ladies. Flowers survives all these troubles,

* 'Alan St Aubyn' was Mrs Frances Marshall,[133] wife of Matthew Marshall.

133 Frances Bridges Marshall (1837–1920): novelist and essayist. (Not to be confused with Frances Catherine Partridge, *née* Marshall (1900–2004).)

134 Marie Corelli (1855–1924): novelist. Corelli's novels, though popular in their time, were generally seen as badly-written melodramas by critics.

† Actually, Chesterton lacks picturesque features.

135 Chesterton is a suburb of Cambridge, northeast of the city itself.

is Second Wrangler[136] and Senior Classic,[137] and succeeds automatically to a Fellowship (as I suppose he would have done then). Brown succumbs, ruins his parents, takes to drink, is saved from delirium tremens during a thunderstorm only by the prayers of the Junior Dean, has much difficulty in obtaining even an Ordinary Degree, and ultimately becomes a missionary. The friendship is not shattered by these unhappy events, and Flowers's thoughts stray to Brown, with affectionate pity, as he drinks port and eats walnuts for the first time in Senior Combination Room.

Now Flowers was a decent enough fellow (so far as 'Alan St Aubyn' could draw one), but even my unsophisticated mind refused to accept him as clever. If he could do these things, why not I? In particular, the final scene in Combination Room fascinated me completely, and from that time, until I obtained one, mathematics meant to me primarily a Fellowship of Trinity.

I found at once, when I came to Cambridge, that a Fellowship implied 'original work', but it was a long time before I formed any definite idea of research. I had of course found at school, as every future mathematician does, that I could often do things much better than my teachers; and even at Cambridge I found, though naturally much less frequently, that I could sometimes do things better than the College lecturers. But I was really quite ignorant, even when I took the Tripos, of the subjects on which I have spent the rest of my life; and I still thought of mathematics as essentially a 'competitive' subject. My eyes were first opened by Professor Love,[138] who taught me for a few terms and gave me my first serious conception of analysis. But the great debt which I owe to him — he was, after all, primarily an applied mathematician — was his advice

136 A 'Wrangler' was one who obtained first-class honours in Part I of the Cambridge Mathematical Tripos. The person who gained the highest mark in a given year was the 'Senior Wrangler', the next highest-placed was the 'Second Wrangler', and so on. Hardy was Fourth Wrangler in 1898, but placed first when he took Part II of the Tripos in 1900 (Snow, 'Foreword', p. 24).

137 The term 'Classic' was analogous to 'Wrangler' but with reference to the Classical Tripos.

138 Augustus Edward Hough Love (1863–1940): applied mathematician.

to read Jordan's[139] famous *Cours d'analyse*; and I shall never forget the astonishment with which I read that remarkable work, the first inspiration for so many mathematicians of my generation, and learnt for the first time as I read it what mathematics really meant. From that time onwards I was in my way a real mathematician, with sound mathematical ambitions and a genuine passion for mathematics.

I wrote a great deal during the next ten years, but very little of any importance; there are not more than four or five papers which I can still remember with some satisfaction. The real crises of my career came ten or twelve years later, in 1911, when I began my long collaboration with Littlewood, and in 1913, when I discovered Ramanujan. All my best work since then has been bound up with theirs, and it is obvious that my association with them was the decisive event of my life. I still say to myself when I am depressed, and find myself forced to listen to pompous and tiresome people, 'Well, I have done one thing *you* could never have done, and that is to have collaborated with both Littlewood and Ramanujan on something like equal terms.'[140] It is to them that I owe an unusually late maturity: I was at my best at a little past forty,[141] when I was a professor at Oxford. Since then I have suffered from that steady deterioration which is the common fate of elderly men and particularly of elderly mathematicians. A mathematician may still be competent enough at sixty, but it is useless to expect him to have original ideas.

It is plain now that my life, for what it is worth, is finished, and that nothing I can do can perceptibly increase or diminish its value. It is very difficult to be dispassionate, but I count it a 'success'; I have had more reward and not less than was due to a man of my particular grade of ability. I have held a series of

139 Marie Ennemond Camille Jordan (1838–1922): engineer and mathematician; noted for his work in group theory and analysis.

140 According to Paul Erdős, Hardy rated mathematicians on the basis of 'pure talent' on a scale of 0 to 100, giving himself a rating of 25, Littlewood 30, Hilbert 80, and Ramanujan 100 (Berndt, *Ramanujan's Notebooks*, p. 14). This illustrates Hardy's 'boast'.

141 Hardy reached the age of forty in 1917.

comfortable and 'dignified' positions, I have had very little trouble with the duller routine of universities. I hate 'teaching', and have had to do very little, such teaching as I have done having been almost entirely supervision of research; I love lecturing, and have lectured a great deal to extremely able classes; and I have always had plenty of leisure for the researches which have been the one great permanent happiness of my life.[142] I have found it easy to work with others, and have collaborated on a large scale with two exceptional mathematicians; and this has enabled me to add to mathematics a good deal more than I could reasonably have expected. I have had my disappointments, like any other mathematician, but none of them has been too serious or has made me particularly unhappy. If I had been offered a life neither better nor worse when I was twenty, I would have accepted without hesitation.

It seems absurd to suppose that I could have 'done better'. I have no linguistic or artistic ability, and very little interest in experimental science. I might have been a tolerable philosopher, but not one of a very original kind. I think that I might have made a good lawyer; but journalism is the only profession, outside academic life, in which I should have felt really confident of my chances. There is no doubt that I was right to be a mathematician, if the criterion is to be what is commonly called success.

My choice was right, then, if what I wanted was a reasonably comfortable and happy life. But solicitors and stockbrokers and bookmakers often lead comfortable and happy lives, and it is very difficult to see how the world is the richer for their existence. Is there any sense in which I can claim that my life has been less futile than theirs? It seems to me again that there is only one possible answer: yes, perhaps, but, if so, for one reason only.

I have never done anything 'useful'. No discovery of mine has made, or is likely to make, directly or indirectly, for good or ill, the least difference to the amenity of the world. I have helped to train other mathematicians, but mathematicians of the same kind as

142 Hardy used precisely this expression to describe mathematics in his presidential address to the Mathematical Association in 1926 (Hardy, 'The Case Against the Mathematical Tripos', p. 66).

myself, and their work has been, so far at any rate as I have helped them to it, as useless as my own. Judged by all practical standards, the value of my mathematical life is nil; and outside mathematics it is trivial anyhow. I have just one chance of escaping a verdict of complete triviality, that I may be judged to have created something worth creating. And that I have created something is undeniable: the question is about its value.

The case for my life, then, or for that of any one else who has been a mathematician in the same sense in which I have been one, is this: that I have added something to knowledge, and helped others to add more; and that these somethings have a value which differs in degree only, and not in kind, from that of the creations of the great mathematicians, or of any of the other artists, great or small, who have left some kind of memorial behind them.

NOTE

Professor Broad and Dr Snow have both remarked to me that, if I am to strike a fair balance between the good and evil done by science, I must not allow myself to be too much obsessed by its effects on war; and that, even when I am thinking of them, I must remember that it has many very important effects besides those which are purely destructive. Thus (to take the latter point first), I must remember (*a*) that the organization of an entire population for war is only possible through scientific methods; (*b*) that science has greatly increased the power of propaganda, which is used almost exclusively for evil; and (*c*) that it has made 'neutrality' almost impossible or unmeaning, so that there are no longer 'islands of peace' from which sanity and restoration might spread out gradually after war. All this, of course, tends to reinforce the case *against* science. On the other hand, even if we press this case to the utmost, it is hardly possible to maintain seriously that the evil done by science is not altogether outweighed by the good. For example, if ten million lives were lost in every war, the net effect of science would still have been to increase the average length of life. In short, my § 28 is much too 'sentimental'.

I do not dispute the justice of these criticisms, but, for the reasons which I state in my preface, I have found it impossible to meet them in my text, and content myself with this acknowledgement.

Dr Snow has also made an interesting minor point about § 8. Even if we grant that 'Archimedes will be remembered when Aeschylus is forgotten', is not mathematical fame a little too 'anonymous' to be wholly satisfying? We could form a fairly coherent picture of the personality of Aeschylus (still more, of course, of Shakespeare or Tolstoi[143]) from their works alone, while Archimedes and Eudoxus would remain mere names.

Mr J. M. Lomas[144] put this point more picturesquely when we were passing the Nelson column in Trafalgar Square. If I had a statue on a column in London, would I prefer the column to be so high that the statue was invisible, or low enough for the features to be recognizable? I would choose the first alternative, Dr Snow, presumably, the second.

143 Lev Nikolayevich Tolstoy [Лев Николаевич Толстой] (1828–1920): novelist and essayist.

144 To whom the *Apology* is dedicated; see p. 2.

MATHEMATICS IN WAR-TIME

The editor asked me at the beginning of term to write an article for EUREKA, and I felt that I ought to accept the invitation; but all the subjects which he suggested seemed to me at the time quite impossible. "My views about the Tripos" — I have never really been much interested in the Tripos since I was an undergraduate, and I am less interested in it now than ever before. "My reminiscences of Cambridge" — surely I have not yet come to that. Or, as he put it, "something more topical, something about mathematics and the war" — and that seemed to me the most impossible subject of all. I seemed to have nothing at all to say about the functions of mathematics in war, except that they filled me with intellectual contempt and moral disgust.

I have changed my mind on second thoughts, and I select the subject which seemed to me originally the worst. Mathematics, even my sort of mathematics, has its "uses" in war-time, and I suppose that I ought to have something to say about them; and if my opinions are incoherent or controversial, then perhaps so much the better, since other mathematicians may be led to reply.

I had better say at once that by "mathematics" I mean *real* mathematics, the mathematics of Fermat and Euler and Gauss and Abel,[1]

1 A modified version of this sentence appears in the *Apology*, § 21, para. 1.

and not the stuff which passes for mathematics in an engineering laboratory. I am not thinking only of "pure" mathematics (though that is naturally my first concern); I count Maxwell and Einstein and Eddington and Dirac among 'real' mathematicians. I am including the whole body of mathematical knowledge which has permanent aesthetic value, as for example, the best Greek mathematics has, the mathematics which is eternal because the best of it may, like the best literature, continue to cause intense emotional satisfaction to thousands of people after thousands of years.[2] But I am not concerned with ballistics or aerodynamics, or any of the other mathematics which has been specially devised for war. That (whatever one may think of its purposes) is repulsively ugly and intolerably dull; even Littlewood could not make ballistics respectable, and if he could not, who can?[3]

Let us try then for a moment to dismiss these sinister by-products of mathematics and to fix our attention on the real thing. We have to consider whether real mathematics serves any purposes of importance in war, and whether any purposes which it serves are good or bad. Ought we to be glad or sorry, proud or ashamed, in war-time, that we are mathematicians?

It is plain at any rate that the real mathematics (apart from the elements) has no direct utility in war. No one has yet found any war-like purpose to be served by the theory of numbers or relativity or quantum mechanics, and it seems very unlikely that anybody will do so for many years. And of that I am glad, but in saying so I may possibly encourage a misconception.

It is sometimes suggested that pure mathematicians glory in the "uselessness" of their subject, and make it a boast that it has no "practical" applications.[*] The imputation is usually based on an

2 Rearranged versions of this sentence and the preceding one appear in the *Apology*, § 25, para. 1.

3 From 'repulsively' onwards, this sentence appears word-for-word in the *Apology*, § 28, para. 3.

* I have been accused of taking this view myself. I once stated in a lecture, which was afterwards printed, that "a science is said to be useful if its development tends to accentuate the existing inequalities in the distribution of wealth, or more directly promotes the destruction of human life"; and this sentence, written

incautious saying attributed to Gauss[*] which has always seemed to me to have been rather crudely misinterpreted. If the theory of numbers could be employed for any practical and honourable purpose, if it could be turned directly to the furtherance of human happiness or the relief of human suffering (as for example physiology and even chemistry can), then surely neither Gauss nor any other mathematician would have been so foolish as to decry or regret such applications. But if on the other hand the applications of science have made, on the whole, at least as much for evil as for good — and this is a view which must always be taken seriously, and most of all in time of war — then both Gauss and lesser mathematicians are justified in rejoicing that there is one science at any rate whose very remoteness from ordinary human activities should keep it gentle and clean.[4]

It would be pleasant to think that this was the end of the matter, but we cannot get away from the mathematics of the workshops so easily. Indirectly, we are responsible for its existence. The gunnery experts and aeroplane designers could not do their job without quite a lot of mathematical training, and the best mathematical training is training in real mathematics.[5] In this indirect way even the best mathematicians becomes important in war-time, and mathematics are wanted for all sorts of purposes. Most of these purposes are ignoble and dreary — what could be more soul-destroying than the numerical solution of differential equations? — but the men chosen for them must be mathematicians and not laboratory hacks, if only because they are better trained and have the better brains. So mathematics is going to be really important now, whether we like it or regret it; and it is not so obvious as it might seem at first

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in 1915, was quoted in the *Observer* only a few months ago. It was, of course, a conscious rhetorical flourish (though one perhaps excusable at the time when it was written).

* To the effect that, if mathematics is the queen of the sciences, then the theory of numbers is, because of its supreme "uselessness", the queen of mathematics. I cannot find an accurate quotation.

4 This paragraph appears, with one introductory sentence and the incorporation of the second footnote in the main text, as *Apology*, § 21, para. 2.

5 This point is made in the *Apology*, § 28, para. 4.

even that we ought to regret it, since that depends upon our general view of the effect of science on war.

There are two sharply contrasted views about modern "scientific" war. The first and the most obvious is that the effect of science on war is merely to magnify its horror, both by increasing the sufferings of the minority who have to fight and by extending them to other classes. This is the orthodox view, and it is plain that, if this view is just, then the only possible defence lies in the necessity for retaliation. But there is a very different view which is also quite tenable. It can be maintained that modern warfare is *less* horrible than the warfare of pre-scientific times, so far at any rate as combatants are concerned; that bombs are probably more merciful than bayonets; that lachrymatory gas and mustard-gas are perhaps the most humane weapons yet devised by military science, and that the "orthodox" view rests solely on loose-thinking sentimentalism. This is the case presented with so much force by Haldane in *Callinicus*.[*] It may also be urged that the equalisation of risks which science was expected to bring would be in the long run salutary; that a civilian's life is not worth more than a soldier's, or a woman's than a man's; that anything is better than the concentration of savagery on one particular class; and that, in short, the sooner war comes "all out" the better.[6] And if this be the right view, then scientists in general and mathematicians in particular may have a little less cause to be ashamed of their profession.

It is very difficult to strike a balance between these extreme opinions, and I will not try to do so. I will end by putting to myself, as I think every mathematician ought to, what is perhaps an easier question. Are there *any* senses in which we can say, with any real confidence, that mathematics "does good" in war? I think I can see two (though I cannot pretend that I extract a great deal of comfort from them).

In the first place it is very probable that mathematics will save the lives of a certain number of young mathematicians, since their technical skill will be applied to "useful" purposes and will keep

* J. B. S. Haldane, *Callinicus*: a defence of chemical warfare (Kegan Paul, 1924).

6 Up to this point, the paragraph is almost identical to *Apology*, § 28, para. 5.

them from the front. "Conservation of ability" is one of the official slogans; "ability" means, in practice, mathematical, physical, or chemical ability; and if a few mathematicians are "conserved" then that is at any rate something gained. It may be a bit hard on the classics and historians and philosophers, whose chances of death are that little much increased; but nobody is going to worry about the "humanities" now. It is better that some should be saved, even if they are not necessarily the most worthy.

Secondly, an older man may (if he is not *too* old) find in mathematics an incomparable anodyne. For mathematics is, of all the arts and sciences, the most austere and the most remote, and a mathematician should be of all men the one who can most easily take refuge where, as Bertrand Russell says, "one at least of our nobler impulses can best escape from the dreary exile of the actual world."[7] But he must not be too old — it is a pity that it should be necessary to make this very serious reservation. Mathematics is not a contemplative but a creative subject; no one can draw much consolation from it when he has lost the power or the desire to create; and that is apt to happen to a mathematician rather soon. It is a pity, but in that case he does not matter a great deal anyhow, and it would be silly to bother about him.[8]

7 Russell, 'The Study of Mathematics', p. 61.

8 This paragraph appears as *Apology*, § 28, para. 7, except that in the *Apology*, the 'incomparable anodyne' can be found by 'a mathematician', not just 'an older man [...] (if he is not *too* old)'.

EDITIONS, EXCERPTINGS, AND TRANSLATIONS

The following lists of editions, excerptings, and translations of *A Mathematician's Apology* and 'Mathematics in war-time' are, to the best of the annotator's knowledge, complete.

Editions of the Apology

First published by Cambridge University Press, 1940; New York: the Macmillan Company, 1940.
Reprinted with C. P. Snow's biographical essay of Hardy (see p. 3, n. 3) as a foreword: Cambridge University Press, 1967.
Canto edition: Cambridge University Press, 1992.

Reprinted in *Great Books of the Western World*, 2nd edition. vol. 56. Encyclopædia Britannica, Inc., 1990.

Excerptings from the Apology

Excerpts (§§ 10–14, 23, 29) reprinted in J. R. Newman, *The World of Mathematics* (vol. 4), pt. XVIII, ch. 1, pp. 2027–38.
Excerpts (§§ 10–11, 22–24) reprinted in Brown, Fauvel & Finnegan, *Conceptions of Inquiry*, § 1.4, pp. 27–31; § 2.3, pp. 50–54.

[Footnotes in § 10 and § 23 omitted.]

Excerpts (§ 12, all but a few sentences of § 13) reprinted in Dawkins, *The Oxford Book of Modern Science Writing*, pt. IV, pp. 352–357.

Translations of the Apology

Apologia d'un matemàtic. Catalan trans. by Mònica Merín i Sales. With an introduction by Josep Pla i Carrera. In: G. H. Hardy *Apologia d'un matemàtic* and J. von Neumann *El paper de la matemàtica en les ciències i la societat*. Obrador Edèndum.

Kēxuéjiā de biànbái [科学家的辩白]. Chinese trans. Nanjing: Jiangsu People's Publishing, 1999.

Yīgè shùxué jiā de biànbái [一个数学家的辩白]. Chinese trans. Beijing: Commercial Press, 2007.

Matemaatikon apologia. Finnish trans. by Kimmo Pietiläinen. Helsinki: Terra Cognita, 1997.

L'Apologie d'un mathématician. French trans. by Dominique Jullien and Serge Yoccoz. In: *Hardy 1877–1947*. Un Savant, une Époque. Paris: Belin, 1985.

Apologia di un matematico. Italian trans. Temi e problemi. Bari: De Donato, 1969.

Apologia di un matematico. Italian trans. by Luisa Saraval. With a preface by Edoardo Vesentini. Garzanti, 2002.

Ichi Sūgakusha no Benmei [一数学者の弁明]. Japanese trans. by Takāki Yagyū. Tokyo: Misuzu, 1975.

Reprinted as *Aru Sūgakusha no Shōgai to Benmei* [ある数学者の生涯と弁明] (*lit.* 'A Mathematician's Life and Apology'). Tokyo: Springer, 1994.

Em Defesa de um Matemático (*lit.* 'In Defense of a Mathematician'). Portuguese trans. by Luís Carlos Borges. São Paulo: Martins Fontes, 2000.

Apologia matematika [*Апология математика*]. Russian trans. by Ju. A. Danilov. Izhevsk: R & C Dynamics, 2000.

Apología de un matemático. Spanish trans. by Jesús Fernández Díez. With a preface by Miguel de Guzman. Epistéme 1. Nivola, 1999.

En matematikers försvarstal. Swedish trans. by Karl-Erik Gustafsson. Lund: Gleerup, 1971.

Bir Matematikçinin Savunması. Turkish trans. by Nermin Arık. Tübitak Yayınları, 2003.

Printings of 'Mathematics in war-time'

First published in *Eureka* 1, no. 3 (January 1940), pp. 5–8.
Reprinted in Hardy, *Collected Papers*, vol. 7, pp. 631–634.
Reprinted in *Eureka* 62 (December 2012), pp. 80–82.
URL: mathigon.org/downloads/eureka-62.pdf.
[Incorporates footnotes into the main text, altering 'only a few months ago' to 'in 1939' in the first and deleting the sentence 'I cannot find an accurate quotation' from the second.]
Reprinted in Albers, Alexanderson & Dunham, *The G.H. Hardy Reader*, pp. 287–290.

CONTEXT OF THE *APOLOGY*

Alan J. Cain

A Mathematician's Apology is the apogee of a tradition that developed in Britain of justifying pure mathematics on aesthetic grounds, a tradition that is interwoven with the very emergence in Britain of pure mathematics as a distinct discipline.[1] Pure mathematics as a discipline effectively did not exist in the Britain at the start of the nineteenth century. Mathematics in Britain was centred on the University of Cambridge, and mathematics there was seen as a part of natural philosophy and was almost entirely isolated from mathematics elsewhere in Europe. Mathematics in Cambridge was of the Newtonian school and emphasized arguments where the meanings in the physical world of the terms employed were to be kept in mind, whereas continental techniques permitted the formal manipulation of symbols. An effort by a group of scholars including Robert Woodhouse, George Peacock, Charles Babbage, and John Herschel introduced the notation and methods

1 Heard, 'The Evolution of the Pure Mathematician' is a full study of this development; the first part of the present account depends upon it.

of continental analysis.[2] The opening of the University of London lessened the dominance of Cambridge over mathematics in Britain.

The foundation of specialist mathematics journals in Britain around the middle of the nineteenth century is an indicator of the emergence of mathematics out of natural philosophy and of the self-recognition of its practioners as researchers in a distinct field.[3] The founding of the London Mathematical Society in 1865 is another.

The pursuit of mathematics in itself, independent of applications, remained debated. George Biddell Airy, the Astronomer Royal from 1835 to 1881, had a great respect for pure mathematics, but only insofar as it could be applied to practical ends, and was averse to mathematical research with no immediate practical application. Arthur Cayley, the first Sadleirian Professor of Pure Mathematics at the University of Cambridge, held that mathematics was a useful mental exercise, and that pure mathematical research could be justified by producing advances that could later be of aid to the sciences. Airy and Cayley debated this point in an exchange of letters in 1867: one of Airy's complaints was that pure mathematicians retreat into isolation and do not contribute to the sciences:

> 'Now as to the Modern Geometry. With your praises of this science […] I entirely agree. And if men, after leaving Cambridge, were designed to shut themselves up in a cavern, they could have nothing better for their subjective amusement. […] But the persons who devote themselves to these subjects do thereby separate themselves from the world. They make no step towards natural science or utilitarian science, the two subjects which the world specially desires. The world could go on as well without these separatists.'[4]

2 Rouse Ball, *A History of the Study of Mathematics at Cambridge*, ch. VII.

3 Heard, 'The Evolution of the Pure Mathematician', ch. 2.

4 Letter to Cayley, 1867-12-09, reprinted in Airy, *Autobiography*, pp. 277–8.

James Joseph Sylvester

Sylvester used part of his 1869 address as president of the mathematics and physical science section of the British Association for the Advancement of Science to justify mathematics as a being based on observation. Although he does not discuss what would later be called platonism in mathematics *per se*, he relates the experience of mathematical observation to that of exploration or discovery in the physical sciences.[5] He offers a defence against an unnamed 'very clever writer' who doubted whether mathematics 'is, in itself, a more serious pursuit, or more worthy of interesting an intellectual human being, than the study of chess problems or Chinese puzzles':[6]

> 'The world of ideas which it discloses or illuminates, the contemplation of divine beauty and order which it induces, the harmonious connexion of its parts, the infinite hierarchy and absolute evidence of the truths with which mathematical science is concerned, these, and such like, are the surest grounds of its title to human regard, and would remain unimpaired were the plan of the universe unrolled like a map at our feet, and the mind of man qualified to take in the whole scheme of creation at a glance.'[7]

Sylvester's remarks appear to be one of the earliest aesthetic *justifications* of the pursuit of mathematics, independently of practical applications. Furthermore, he holds pure mathematics is about something, a 'world of ideas', and is not just an intellectual game.

There is *anecdotal* evidence of a perhaps cynical emergence, around this time, of the view that beauty and utility were opposed. E. W. Hobson recalled 'a very great Pure Mathematician', whom he does not identify, but who would have been at Cambridge in the 1870s, saying that 'Bessel's functions are very beautiful functions, in spite of their having practical applications'.[8] (One speculates

5 Sylvester, 'Presidential Address', pp. 655–7.

6 *Ibid.*, p. 658.

7 *Ibid.*, pp. 658–9.

8 Hobson, *Mathematics*, pp. 4–5.

that hearing such views during his undergraduate years shaped Hobson's own view that utility detracts from the beauty of mathematics: in 1912, he said that number theory had 'never been soiled by any practical application' but wondered whether it would 'always remain undefiled'.[9]) C. H. Pearson reported that H. J. S. Smith once concluded a lecture by saying: '"It is the peculiar beauty of this method, gentlemen," [...] "and one which endears it to the really scientific mind, that under no circumstances can it be of the smallest possible utility."'[10]

It has been argued that the development of aesthetic justifications for pure mathematics, and of the characterization of the pure mathematician as a creative artist, was made plausible by the attitude to art espoused by Walter Pater, which separated moral truth in art from beauty:[11]

> '"To see the object as in itself it really is," has been justly said to be the aim of all true criticism whatever, and in aesthetic criticism the first step towards seeing one's object as it really is, is to know one's own impression as it really is, to discriminate it, to realise it distinctly. [...] What is this song or picture, this engaging personality presented in life or in a book, to *me*? What effect does it really produce on me? Does it give me pleasure? and if so, what sort or degree of pleasure? How is my nature modified by its presence, and under its influence?'[12]

This also makes it clear that beauty can be found *anywhere*, depending on the individual. Furthermore, contemplation of beauty is one of the highest aims of life, for it helps us to fulfil ourselves in the time that we are given:

> 'High passions give one this quickened sense of life, ecstasy and sorrow of love, political or religious enthusiasm, or

9 Hobson, *Mathematics*, p. 13.
10 Pearson, 'Biographical Sketch', pp. xxxiii–xxxiv.
11 Heard, 'The Evolution of the Pure Mathematician', pp. 229–35.
12 Pater, *Studies in the History of the Renaissance*, p. viii; emphasis in original.

> the "enthusiasm of humanity". Only, be sure it is passion, that it does yield you this fruit of a quickened, multiplied consciousness. Of such wisdom, the poetic passion, the desire of beauty, the love of art for art's sake, has most; for art comes to you professing frankly to give nothing but the highest quality to your moments as they pass, and simply for those moments' sake.'[13]

Arthur Cayley

That this view of art was 'in the air' and accepted (or at least accept*able*) is evidenced by Cayley's 1883 presidential address to the British Association for the Advancement of Science. *Addressing a general audience*, Cayley said that if he were to justify pure mathematics

> 'I should desire to do it […] *not* by speaking to you of the utility of mathematics in any of the questions of common life or of physical science. Still less would I speak of this utility before, I trust, a friendly audience, interested or willing to appreciate an interest in mathematics in itself and for its own sake. I would, on the contrary, rather consider the obligations of mathematics to these different subjects as the sources of mathematical theories'[14]

Cayley supported the idea (and here assumed his audience would also) that mathematics can be pursued independent of any applications, although he retained an interest in applications.[15] The physical sciences were a source of inspiration for the development of mathematics, which would in time give back to those theories. However, Cayley held to a platonic view of mathematics:

> 'I would myself say that the purely imaginary objects are the only realities, the ὄντως ὄντα, in regard to which the

13 Pater, *Studies in the History of the Renaissance*, pp. 212–13.
14 Cayley, 'Address', pp. 4–5.
15 Craik, *Mr Hopkins' Men*, p. 333.

corresponding physical objects are as the shadows in the cave [...] at any rate the objects of geometrical truth are the so-called imaginary objects [...], and the truths of geometry are only true, and *a fortiori* are only necessarily true, in regard to these so-called imaginary objects'.[16]

Thus whatever pure mathematics may draw from the physical sciences, it is then not only pursued for completely independent reasons, but it is *about* something completely different from the physical sciences.

Cayley's platonism naturally fitted into his 'Whig history' view of pure mathematics, where nothing is ever lost and there is steady progress. This view is evident from his application to mathematics of the words of Tennyson:[17]

> 'Yet I doubt not thro' the ages one increasing purpose runs,
> And the thoughts of men are widen'd with the process of
> the suns.'[18]

Salmon, writing in the same year, characterized Cayley's life as almost a kind of asceticism: Cayley, he said,

> 'has had courage to despise the allurements of avarice or ambition, and has found more happiness from a life devoted to the contemplation of beauty and truth'.[19]

Certainly Cayley's acceptance of the Sadleirian chair of mathematics entailed a reduction in his income: before this, he had been a successful lawyer. Salmon placed mathematics between the arts and the applied sciences[20] but explicitly called Cayley 'a great artist',[21] and said that his work, like the work of other mathematicians, should not be judged based on the numbers that can appreciate

16 Cayley, 'Address', p. 7.

17 *Ibid.*, p. 37.

18 Tennyson, 'Locksley Hall', ll. 137–8; Cayley expanded the apostrophized words.

19 Salmon, 'Arthur Cayley', p. 483.

20 *Ibid.*, p. 484.

21 *Ibid.*, p. 483.

it, just as artists are not so judged. Heard suggested that Salmon's portrayal of Cayley marked a turning-point when pure mathematicians started to become more confident in justifying the pursuit of pure mathematics independent of any applicability.[22]

J. W. L. Glaisher

Glaisher seemed to agree with the asceticism that Salmon ascribed to Cayley. At one point he dismissed the idea that money could influence pure mathematicians towards pursuing particular research: he complained about a prize offered on the wrappers of a volume of the *American Journal of Mathematics*, edited by Sylvester, for the proof or disproof of a particular conjecture. He dismissed this as an 'anachronism', and said that:

> 'It seems unlikely that any competent person would be tempted to investigate the subject by hope of the reward. Pure mathematics offers no mercenary inducements to its followers, who are attracted to it by the importance and beauty of the truths it contains; and the complete absence of any material advantage to be gained by means of it, adds perhaps even another charm to its study.'[23]

Note that the normative claim is about 'any competent person': those who would pursue financial gain through mathematics are thus not of the highest intellectual calibre. First-rate minds seek in mathematics values that include beauty but exclude material gains.

Glaisher, however, has been called a transitional figure, in that he completely accepted that pure mathematics could and should be pursued for its own sake, but that he sometimes wavered towards applied mathematics being a worthier calling than pure.[24] In 1890, as president of the mathematical and physical sciences section of the British Association for the Advancement of Science, he said

22 Heard, 'The Evolution of the Pure Mathematician', p. 249.
23 Glaisher, '*American Journal of Mathematics, Pure and Applied*', p. 195.
24 Heard, 'The Evolution of the Pure Mathematician', p. 128.

that every mathematician cherished the hope that their field, however recondite, would find a practical application; yet, in the same address, he said that that pure mathematics could not be justified on the basis of its applications.[25] He regretted that mathematical training at Cambridge was so focussed on applications,[26] and said that:

> 'it always appears to me that there is a certain perfection, and also a certain luxuriance and exuberance, in the pure sciences [...] which is conspicuously absent from most of the investigations which have had their origin in the attempt to forge the weapons required for research in the less abstract sciences'.[27]

Yet he also rejected the idea that researchers should be trained only in pure mathematics, so that only specialists could pursue the topic.[28] The only reason for a researcher to pursue pure mathematics is an aesthetic one: 'no one should devote himself to the abstract sciences unless he feels strongly drawn to them by his tastes'.[29]

These various aesthetic justifications for pure mathematics have a commonality in that they are all very individualistic. When the *mathematician* pursues beauty for its own sake, the reward is to the mathematician themself. Of course, others may read and appreciate the beautiful mathematics thus produced, but this is not stated as a motivation. The closest approach to a social justification for pure mathematics is an observation made by Glaisher:

> 'The search after abstract truth for its own sake, without the smallest thought of practical applications or return in any form, and the yearning desire to explore the unknown, are signs of the vitality of a people, which are among the first to disappear when decay begins.'[30]

25 Glaisher, 'Presidential address', pp. 722–3.
26 *Ibid.*, p. 721.
27 *Ibid.*, p. 720.
28 *Ibid.*, pp. 724–5.
29 *Ibid.*, p. 725.
30 *Loc. cit.*

Note, though, that this is not in itself a justification: a healthy society values the pursuit by some of its members of pure mathematics, but it does not follow that a society can be made healthy by encouraging such research. (Compare the much later assertion by the historian G. M. Trevelyan, Master of Trinity College when the *Apology* was published: 'Disinterested intellectual curiosity is the life-blood of real civilization.'[31])

Henri Poincaré

For the mathematician, physicist, and philosopher Henri Poincaré (1854–1912), beauty was a motivation in the sciences generally. Writing in 1905, he explicitly made the point that, as a motivation, beauty was more important than utility in science:

> 'The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living.'[32]

Furthermore, he held that focussing research purely on utility would be counterproductive in terms of the production of scientific results and the coherence of the theory.[33]

As regards mathematics specifically, Poincaré thought that there are three interconnected aims in doing mathematics:

> 'They must furnish an instrument for the study of nature. But that is not all: they have a philosophic aim and, I dare maintain, an esthetic aim. They must aid the philosopher to fathom the notions of number, of space, of time. And above all, their adepts find therein delights analogous to those given by painting and music.'[34]

31 Trevelyan, *English Social History*, p. viii.
32 Poincaré, *The Value of Science*, Preface.
33 Poincaré, 'Analysis and Physics', § 1.
34 *Loc. cit.*

It is for these reasons that Poincaré held that mathematics should be studied for its own sake, including areas inapplicable to physics.[35] However, Poincaré took a position opposite to that later expressed by Hardy in the *Apology* about the position of applied versus pure mathematicians: while Hardy found that the imaginary universes of pure mathematics more beautiful than physical reality,[36] Poincaré averred that the superior aesthetic value of physical reality meant that the scientist would not be distracted from their quest: 'One may dream a harmonious world, but how far the real world will leave it behind!'.[37]

Poincaré discussed both beauty and elegance in mathematics;[38] it is unclear whether this was a deliberate distinction between two kinds of aesthetic value. In a proof or solution, the perception of elegance is produced by

> 'the harmony of the different parts, their symmetry, and their happy adjustment; it is, in a word, all that introduces order, all that gives them unity, that enables us to obtain a clear comprehension of the whole as well as of the parts. [...] Elegance may result from the feeling of surprise caused by the unlooked-for occurrence together of objects not habitually associated.'[39]

The feeling of elegance is the emotional response to some parallel between the solution before us and 'the necessities of our mind', and it is this parallel that helps us to *use* the solution: 'aesthetic satisfaction is consequently connected with the economy of thought'.[40] Thus Poincaré linked the aesthetic appeal of mathematics to its usefulness, at least via the indirect development of other mathematics that can then provide 'an instrument for the study of nature'.

35 Poincaré, 'Analysis and Physics', § 1.
36 Hardy, *Apology*, § 26.
37 Poincaré, *The Value of Science*, Preface.
38 Poincaré, 'The Future of Mathematics'; Poincaré, 'Mathematical Discovery'.
39 Poincaré, 'The Future of Mathematics'.
40 *Ibid.*

Hardy was certainly aware of Poincaré's essay on mathematical discovery, for he cited it in his 1946 review of Jacques Hadamard's *An Essay on the Psychology of Invention in the Mathematical Field*,[41] but it is unclear whether he had read it by 1940, when he wrote the *Apology*, or whether he had read Poincaré's other essays that consider aesthetics in mathematics and science. Nevertheless, Poincaré's philosophical work would not be out of place as a precursor to Hardy: Poincaré's views of motivation fitted with Hardy's, though he did not seem so taken as Hardy with mathematical as opposed to physical reality; doubtless this is to be expected given Poincaré's generally recognized universalism in mathematics and physics.

G. E. Moore

Moore argued in his *Principia Ethica* that 'goodness' is indefinable,[42] and that to attempt to define it in terms of concepts such as 'desire' or 'pleasure' is to commit the *naturalistic fallacy*.[43] Moore held that the naturalistic fallacy can be committed in aesthetic as well as ethical reasoning:[44] beauty, like goodness, is indefinable. Nevertheless, judgements of both beauty and goodness are objectively true or false. The objective Moorean view of judgements of beauty would harmonize with Hardy's platonic view of mathematics. Furthermore, experiencing beauty is a fundamental good:

> 'By far the most valuable things, which we know or can imagine, are certain states of consciousness, which may be roughly described as the pleasures of human intercourse and the enjoyment of beautiful objects.'[45]

It is uncertain whether Hardy actually read the *Principia Ethica*, but he would certainly have been aware of it: both men were fellows

41 Hardy, 'Review of *The Psychology of Invention*', p. 111.
42 Moore, *Principia Ethica*, §§ 6 sqq.
43 *Ibid.*, §§ 10–14.
44 *Ibid.*, § 121.
45 *Ibid.*, § 113.

of Trinity College, and Hardy was associated with the Bloomsbury Group,[46] for whom Moore and the *Principia Ethica* held great value.[47] In any case, it forms part of the intellectual background against which the *Apology* was written, and would have lent credence to the idea that a good life can be devoted to the pursuit of beauty.

Bertrand Russell

In his 1907 essay 'The Study of Mathematics', Russell analysed aesthetics both of proofs and of theories. Hardy must have read this essay, because he quoted from it in § 28 of the *Apology* (although he only attributed the quotation to Russell, without giving the source). This is therefore the earliest work that is explicitly a precursor of the *Apology*, and there are clear parallels between them. This does not immediately imply that Hardy's thinking was shaped by 'The Study of Mathematics', for Hardy and Russell had a similar background mathematically and perhaps the parallels are the result of drinking at the same founts of knowledge.[48] That said, in philosophy of mathematics, Hardy was a follower of Russell[49] and kept abreast of at least the outlines of his work: he reviewed *Principles of Mathematics*[50] and the first volume of *Principia Mathematica*,[51] and several of his other book reviews contain references to Russell's philosophy.[52] His obituarist Titchmarsh went so far as to call him a 'disciple' of Russell.[53]

For Russell, the most beautiful mathematical proofs were such that the

46 Snow, 'Foreword', p. 25.
47 Rosenbaum, *Edwardian Bloomsbury*, p. 3 *&* *passim*.
48 Grattan-Guinness, 'Russell and G.H. Hardy'.
49 Grattan-Guinness, 'The interest of G.H. Hardy', pp. 412–15.
50 Hardy, 'Review: The Principles of Mathematics'.
51 Hardy, 'Review of *Principia Mathematica*'.
52 Hardy, 'Review: A New Algebra'; Hardy, 'Review *The Theory of the Imaginary*'.
53 Titchmarsh, 'Godfrey Harold Hardy', p. 84.

> ‘chain of argument is presented in which every link is important on its own account, in which there is an air of ease and lucidity throughout, and the premises achieve more than would have been thought possible, by means which appear natural and inevitable.’[54]

Further, in the reasoning, ‘unity and inevitability are felt as in the unfolding of a drama’.[55] Such phrases bring to mind Hardy’s ‘“purely aesthetic” qualities’ of theorems and proofs in the *Apology*.[56] Although Hardy did not use the term ‘unity’, it is clearly implicit in his deprecation of enumeration of cases.

Russell held that mathematics is motivated by a search for beauty. Elsewhere he divided motivations into two groups, *possessive* and *creative*. The possessive impulse aims to acquiring sole possession of something; the creative tries to give something valuable to the world. He considered ‘the best life that which is most built on creative impulses, and the worst that which is most inspired by love of possession.’[57] The drive toward discovery, and in particular mathematical discovery, is clearly a creative impulse, and the knowledge it aims to deliver to the world is valuable. The value may lie partly in application, but also in beauty. In particular, in pure mathematics one is not limited to what is applicable to the world: one can give free rein to

> ‘reason’s privilege of dealing with whatever objects its love of beauty may cause to seem worthy of consideration.’[58]

For the mathematician, the idea that the mathematics they discover may some day be useful can be a comfort in times of doubt. It cannot, however, be a guide in their research: for example, the study of conic sections in antiquity was pursued without any glimmering that eighteen centuries later they would be used by

54 Russell, ‘The Study of Mathematics’, p. 61.
55 *Ibid.*, p. 66.
56 Hardy, *Apology*, § 18.
57 Russell, *Principles of Social Reconstruction*, p. 5.
58 Russell, ‘The Study of Mathematics’, p. 70.

Kepler in formulating his laws of planetary motion.[59] However, the study of mathematics does have a worthwhile effect in helping to inculcate 'a lofty habit of mind'.[60] (Here, perhaps, is a shade of Plato's prescription of mathematical education for the rulers in the *Republic*.[61])

Another motivation for mathematics is that, on an individual level, it can serve as a refuge from the troubles of the world, whither 'our nobler impulses can escape'.[62] This is the part of Russell's essay that Hardy quoted,[63] when he called mathematics an 'incomparable anodyne'. Russell was not unique in holding this view. In 1918, Einstein said:

> 'I believe with Schopenhauer that one of the strongest motives that leads men to art and science is escape from everyday life with its painful crudity and hopeless dreariness, from the fetters of one's own ever shifting desires. A finely tempered nature longs to escape from personal life into the world of objective perception and thought'[64]

In citing Schopenhauer, Einstein, like Russell and later Hardy, linked mathematics as a refuge to mathematics as an art. In brief, Schopenhauer held that life is suffering,[65] but he identified ways in which it is possible to create more peaceful states of mind. Aesthetic perception is a means of reaching such a state,[66] and in particular contemplation of the beautiful leads to an easy transition to this state.[67]

Russell did wonder whether is it ethical to devote oneself to mathematics, guided by 'love of beauty':

59 Russell, 'The Study of Mathematics', p. 72.
60 *Ibid.*, p. 73.
61 For a study, see Burnyeat, 'Plato on Why Mathematics is Good for the Soul'.
62 Russell, 'The Study of Mathematics', p. 70.
63 Hardy, *Apology*, § 28.
64 Einstein, 'Principles of Research', p. 225.
65 Schopenhauer, *The World as Will and Representation*, vol. 1, bk 4, § 56.
66 *Ibid.*, vol. 1, bk 3, § 34.
67 *Ibid.*, vol. 1, bk 3, § 39.

> 'In a world so full of evil and suffering, retirement into the cloister of contemplation, to the enjoyment of delights which, however noble, must always be for the few only, cannot but appear as a somewhat selfish refusal to share the burden imposed upon others by accidents.'[68]

His answer was twofold: 'some must keep alive the sacred fire',[69] and the fact, already mentioned, that there is no way of telling in advance which parts of mathematics will prove useful.

Russell's defence of mathematics stood on the cusp between individual and social. On the one hand, pursuing the mental states that beauty creates — '[t]he true spirit of delight, the exaltation, the sense of being more than man, which is the touchstone of the highest excellence'[70] — and taking refuge from the world in mathematics are individual motivations. On the other hand, the unknowable potential utility of each piece of mathematics is ultimately an appeal to a social end. Beauty can be both an individual and a social end, the latter because the creative impulse, for Russell, meant giving something to the world. Even maintaining the 'sacred fire' is ultimately a social end, albeit a rather mystical one.

War and Aftermath

In 1915, Hardy lectured on number theory to the British Association for the Advancement of Science.[71] In a passage which he later discussed in the *Apology*,[72] he defined a useful science as one that 'tends to accentuate the existing inequalities in the distribution of wealth, or more directly promotes the destruction of human life.'[73] In a sustained irony, he said that he could not defend number theory and in particular the theory of primes on this ground, and indeed that a wise person would not even attempt to

68 Russell, 'The Study of Mathematics', p. 72.
69 *Loc. cit.*
70 *Ibid.*, p. 60.
71 Hardy, 'Prime Numbers'.
72 Hardy, *Apology*, § 21.
73 Hardy, 'Prime Numbers', p. 350.

justify their interest in such a subject. This was an understandable reaction, given that when Hardy spoke in September 1915 the First World War had clearly become an industrial war and had already witnessed the use of chemical weapons, notably at the the Second Battle of Ypres.

Yet, for precisely the reason of its value in war, science began to be assigned a higher value by society in the Interbellum. J. W. N. Sullivan noted that mathematics also benefitted from this higher value placed on the sciences in the aftermath of the First World War.[74] But he constrasted this with the view mathematicians held of their own field: he accepted the assertion of mathematicians that their field is a 'delightful one'[75] *for them* and 'that the mathematicians are impelled by the same incentives and experience the same satisfactions as other artists',[76] but held that this is insufficient justification: there is no reason that society should support a pleasure obtainable by so few. While Hardy considered chess a form of mathematics,[77] Sullivan explicitly contrasts chess with mathematics: 'Chess professorships are not established, but there are probably more people who appreciate the "beauties" of chess than appreciate the beauties of mathematics.'[78]

A certain amount of anecdotal evidence points to broad acceptance that utility and applications were disdained in pure science generally between the wars. C. P. Snow, in his famous 1959 Rede lecture *The Two Cultures*, said that:

> 'Pure scientists have by and large been dim-witted about engineers and applied science. […] Their instinct […] was to take it for granted that applied science was an occupation for second-rate minds. I say this more sharply because thirty years ago I took precisely that line myself. […] We prided ourselves that the science we were doing could not,

74 Sullivan, 'Mathematics as an Art', p. 2015.

75 *Loc. cit.*

76 *Ibid.*, p. 2020.

77 Hardy, *Apology*, § 10.

78 Sullivan, 'Mathematics as an Art', p. 2015.

> in any conceivable circumstances, have any practical use. The more firmly one could make that claim, the more superior one felt.'[79]

The classical scholar F. M. Cornford, who, like Hardy, was a fellow of Trinity College, said in a popular lecture that

> 'science as commonly defined [is] the pursuit of knowledge for its own sake, not for any practical use it can be made to serve.'[80]

The defence that Glaisher and Russell had offered for pure mathematics — that it is impossible to forecast what parts of mathematics will be useful — was offered again as a defence of pure science and curiosity more generally by Abraham Flexner in 1939 in his provocatively-titled essay 'The usefulness of useless knowledge': 'the pursuit of these useless satisfactions proves unexpectedly the source from which undreamed-of utility is derived'[81] A slightly different justification, encompassing the notion that it is impossible to forecast which areas of pure science will lead to harmful and which to beneficial applications, was given by Lord Rayleigh in his presidential address to the British Association for the Advancement of Science in 1938.[82]

E. T. Bell held as a truth of experience that, for whatever reason, mathematics pursued for aesthetic ends turns out to be useful in the natural sciences.[83] He argued in favour of allowing mathematicians to follow their own interests, for experience tends to show that whatever mathematicians study will turn out to be vital to science and industry in the future;[84] he even went so far as to counsel against directing mathematical research towards immediate applications:

79 Snow, *The Two Cultures*, § 3.
80 Cornford, *Before and After Socrates*, p. 5.
81 Flexner, 'The usefulness of useless knowledge', p. 544.
82 Rayleigh, 'Presidential Address', esp. p. 30.
83 Bell, *The Queen of the Sciences*, pp. 1–3.
84 *Ibid.*, pp. 81–84.

> 'Guided only by their feeling for symmetry, simplicity, and generality, and an indefinable sense of the fitness of things, creative mathematicians now as in the past are inspired by the art of mathematics rather than by any prospect of ultimate usefulness. However it may be in engineering and the sciences, in mathematics the deliberate attempt to create something of immediate utility leads as a rule to shoddy work of only passing value. The important practical and scientific applications of mathematics are unsought byproducts of the main purposes of professional mathematicians.'[85]

The first two sentences in this passage seem positively Hardian, but Bell only defends the aesthetic motivation for mathematics as a means to the end of utility. Thus there is no contradiction between his views here and his later sardonically critical review of the *Apology* (see page 98).

Lancelot Hogben, whose *Mathematics for the Million* was a bugbear for Hardy, connected mathematical value to applicability in an entirely different way. Hogben's view is that mathematics only progresses in line with its applications:

> 'mathematics has advanced when there has been real work for the mathematician to do, and that it has stagnated whenever it has become the plaything of a class which is isolated from the common life of mankind'[86]

Hogben seems to assign value to mathematics only (or, at least, almost entirely) in terms of its applications. Although he does not say so explicitly, one could infer that, for him, an advance in mathematics that does not have an application is not an advance in what he would see as 'real mathematics'.

85 Bell, *The Queen of the Sciences*, pp. 2–3.
86 Hogben, *Mathematics for the Million*, p. 36.

Wolfgang Krull

Krull's 1930 inaugural lecture at the University of Erlangen was dedicated to explaining his own personal view that the imagination and creativity of the mathematician and the artist are closely related,[87] but his aim and argument were rather different from those of the authors discussed above. The most important difference is that he did not explicitly offer a defence of or justification for mathematics, although some of his language — 'a personal confession of faith'[88] — seems to find a faint echo in the *Apology*. He did imagine his listeners thinking '"Until now we always thought that the ultimate goal of mathematics was its application to practical problems. Now we see that [...] the major role is played by so-called aesthetic considerations. [...] Is it worth anything at all?"'[89] But his answer, if it can be called that, was to lament the isolation of mathematicians and their work, and to hope that, just as the history of mathematics shows it has become easier to understand, so it will continue to become more accessible in future.

The other major difference is that Krull argued for a parallel between mathematical and artistic imagination by emphasizing that a kind of mathematical beauty can be found in visual art. He paraphrased the group theorist Andreas Speiser[90] in claiming a close connection between the visual aspect of a tiling pattern and its underlying symmetry group, and supported this with the example of a design approximating a logarithmic spiral carved on a Viking ship. He held that the aesthetic value of the carved spiral is due to the group of scaling-plus-rotation transformations that preserve the spiral: 'I believe that it is precisely the mathematical group behind the spiral that is responsible for its aesthetic value.'[91]

87 Krull, 'The aesthetic viewpoint in mathematics', pp. 48, 49.
88 *Ibid.*, p. 48.
89 *Ibid.*, p. 52.
90 See, for example, Speiser, *Die Theoerie der Gruppen von Endlicher Ordnung*.
91 Krull, 'The aesthetic viewpoint in mathematics', p. 49.

Krull thought that '[a] beautiful ornament should indeed set before the viewer an especially striking presentation of the totality of properties of the underlying group'.[92] This visual beauty of the physical artwork does not only reflect the underlying mathematical structure, but can motivate the creation of elegant mathematics through the investigation of that structure.[93] Further, Krull hinted at a deep connection between the aesthetic value of an ornament, and the aesthetic value in results and proofs: on the one hand, he said '[m]athematicians [...] want to arrange and assemble the theorems so that they appear not only correct but evident and compelling. Such a goal, I feel, is aesthetic rather than epistemological.'[94] On the other hand, 'a mathematician may find that an appropriate ornament is the most attractive way to present the mathematics'.[95]

Beauty in mathematics may sometimes be achieved at the cost of complete rigour. Krull gave the example of Felix Klein's work, which has a 'particularly captivating charm',[96] due to his emphasis on geometric, visual reasoning, which is expressed through figures that 'illustrate the underlying mathematical relationships in an extremely simple and transparent way',[97] but which nevertheless contains flawed proofs. This does not mean that elegance should be prized above rigour. Yet there is a question of balance between rigour and elegance:

> 'A mathematician who is concerned above all with the irrefutable certainty of his results will try to base his theorems on as few unproved assumptions as possible. Consequently he will only feel secure in geometry, for instance, when he has completely reduced that subject to arithmetic [...]. [H]e will even try to reduce the system of whole numbers to something even simpler, perhaps a system of logic. In

92 Krull, 'The aesthetic viewpoint in mathematics', p. 49.

93 *Loc. cit.*

94 *Loc. cit.*

95 *Loc. cit.*

96 *Ibid.*, p. 50.

97 *Loc. cit.*

> short, he will devote himself to what people in mathematics nowadays call the study of foundations.
>
> The more aesthetically oriented mathematician will have less interest in the study of foundations, with its painstaking and often necessarily complicated and unattractive investigations. He will of course unfailingly fit his proofs to the rigor of his time, but he will not rack his brains about whether his theorems are proved in a way that will necessarily be considered absolutely flawless under all conditions for all eternity.'[98]

In this single lecture, Krull set out a complex and nuanced position, in some ways an outlier, as to the value of mathematics. He admitted and regretted its remoteness. He celebrated beauty and admitted it as motive without proclaiming it a defence. He accepted a tension between beauty and rigour. He posited a connection between visual beauty of certain artworks and mathematical beauty. He admitted that practical applications did not motivate mathematicians. Finally, he did not even attempt, like other authors of the 1920s and 1930s, to defend mathematics on the grounds that there is no way to tell from which topics applications may arise in future.

G. H. Hardy

Except for Russell's 'The Study of Mathematics', which is referenced in § 28, it is unknown whether Hardy had read any of the aesthetic justifications for mathematics discussed above when he wrote the *Apology*. However, it is reasonable to think that the views expressed in them were common enough among pure mathematicians to have influenced Hardy's view of the value of mathematics during his formative years and professional life.

As regards Hardy's own writings, the *Apology* is not an *ex nihilo* discussion of beauty, for aesthetic concerns appear regularly

98 Krull, 'The aesthetic viewpoint in mathematics', p. 50.

in Hardy's writing throughout his career. For example, Hardy frequently used aesthetic terms to evaluate books he reviewed and the mathematics of the subjects of obituaries he wrote.[99] To take a single illustrative example, Hardy's 1921 obituary of Ramanujan distinguishes the time when Ramanujan produced 'some of his most beautiful theorems',[100] and mentions that 'it would be difficult to find more beautiful formulæ'[101] than the Rogers–Ramanujan identities:

$$1 + \frac{q}{1-q} + \frac{q^4}{(1-q)(1-q^2)} + \frac{q^9}{(1-q)(1-q^2)(1-q^3)} + \dots$$
$$= \frac{1}{(1-q)(1-q^4)(1-q^6)(1-q^9)(1-q^{11})(1-q^{14})\cdots},$$
$$1 + \frac{q^2}{1-q} + \frac{q^6}{(1-q)(1-q^2)} + \frac{q^{12}}{(1-q)(1-q^2)(1-q^3)} + \dots$$
$$= \frac{1}{(1-q^2)(1-q^3)(1-q^7)(1-q^8)(1-q^{12})(1-q^{13})\cdots}.$$

These identities were first discovered and proved by L. J. Rogers[102] as corollaries of general results. Ramanujan independently rediscovered them but had no proof. Rogers later supplied another proof, and in 1919 he and Ramanujan independently gave much simpler proofs using similar principles, which Hardy described as 'the simplest and most elegant proofs'.[103] However, Hardy did not consider any of these proofs to be truly '"simple" and "straightforward"' and said 'no doubt it would be unreasonable to expect a really easy proof'.[104] Here is an indication that, for Hardy, a result could possess beauty independent of proof.

Hardy's reviews also supply some insight into his views on the aesthetics of mathematical *theories*. He took pleasure in developments that allowed the reconstitution of theories into a more

99 See Hardy, *Collected Papers*, vol. 7, pts 4–5.
100 Hardy, 'Srinivasa Ramanujan', p. xxxi.
101 *Ibid.*, p. xxxiv.
102 Rogers, 'Second Memoir', § 5.
103 Rogers & Ramanujan, 'Proof of certain identities', Preface by Hardy.
104 Hardy, *Ramanujan*, p. 91.

elegant form. An exemplary case was the work of Borel and then of Lebesgue, which led to the rewriting of the previously inelegant theory of integrals.[105] He explained how the inelegance of a theory can lead to a reformulation as part of a larger, more general, theory; he illustrated his thesis with the examples of extending the rational numbers to the reals in analysis, and extending real to complex geometry. In both cases, the original theory had become 'honeycombed with exceptions and distinctions until it has become aesthetically intolerable',[106] so that one desired to banish the anomalies.

However, whatever Hardy may have said in the *Apology*, Mordell (Hardy's successor as Sadleirian Professor of Pure Mathematics at Cambridge) found that Hardy's research was not marked by an enduring quest for beauty, but was

> 'distinguished more by his insight, his generality, and the power he displays in carrying out his ideas. […] the proofs are often long and require concentrated attention, and this may blunt one's feelings even if the ideas are beautiful.'[107]

Pólya's view fitted with Mordell's:

> 'Hardy wrote very well and with great facility, but his papers […] make no easy reading: The problems are very hard and the methods unavoidably very complex. He valued clarity, yet what he valued most in mathematics was not clarity but power, surmounting great obstacles that others abandoned in despair.'[108]

The *Apology* was written when Hardy was confronted with decline. During the 1930s, Hardy had continued to play real tennis and squash, but a coronary thrombosis in 1939 ended such physical

105 See, for example, Hardy, 'Prof. H.L. Lebesgue', p. 685 and Hardy, 'Review of *The Theory of Functions*'.

106 Hardy, 'Review *The Theory of the Imaginary*', p. 78.

107 Mordell, 'Hardy's "A Mathematician's Apology"', p. 834.

108 Pólya, 'Some mathematicians', p. 751.

activities.[109] An awareness of his fading creative powers suffuses the *Apology* and gives it what C. P. Snow called its 'haunting sadness'.[110] But Hardy was equally vocal about the joys of mathematics during the 1920s:

> 'It is hardly likely that anybody here will accuse me of any lack of devotion to the subject which has after all been the one great permanent happiness of my life. My devotion to mathematics is indeed of the most extravagant and fanatical kind; I believe in it, and love it, and should be utterly miserable without it, and I have never doubted that, for any one who takes real pleasure in it and has a genuine talent for it, it is the finest intellectual discipline in the world.'[111]

[Hardy re-used the expression 'the one great permanent happiness of my life' in the *Apology*.[112]]

The genesis of the *Apology* came when the editor of *Eureka*, the journal of the Archimedean Society (the University of Cambridge undergraduate mathematical society), invited Hardy to contribute an article. Hardy wrote that the invitation was made 'at the beginning of term';[113] since the article appeared in the January 1940 issue, this presumably refers to the start of Michaelmas Term in October 1939. Hardy initially rejected the ideas that he should give his views of the Mathematical Tripos, or reminisce about Cambridge, or write '" [...] something about mathematics and the war"'.[114] The last suggestion seemed to him the worst, for the uses of mathematics in war provoked in Hardy 'intellectual contempt and moral disgust'.[115] But after this initial hesitation, this was the subject he chose to write on, and the resulting essay was 'Mathematics in war-time' (see pages 64 ff.).

109 Snow, 'Foreword', p. 50.
110 *Loc. cit.*
111 Hardy, 'The Case Against the Mathematical Tripos', p. 66.
112 Hardy, *Apology*, § 29.
113 Hardy, 'Mathematics in war-time'.
114 Hardy, 'Mathematics in war-time'.
115 Hardy, 'Mathematics in war-time'.

The title Hardy chose for this essay echoes that of C. S. Lewis's sermon 'Learning in war-time', preached in the University Church of St Mary the Virgin in Oxford on 22 October 1939 and distributed in print under the titles '"None Other Gods": Culture in War-Time' and 'The Christian in Danger' during the time Hardy was writing.[116] Part of Lewis's argument was that 'useless' or 'disinterested' cultural life can, will, and must continue during war:

> 'Men [...] propound mathematical theorems in beleaguered cities, conduct metaphysical arguments in condemned cells, make jokes on scaffolds, discuss the last new poem while advancing to the walls of Quebec, and comb their hair at Thermopylae. This is not *panache*; it is our nature.'[117]

And for Lewis, such intellectual acts can be spiritual if they are humbly directed to God and the impulse for them is kept 'pure and disinterested'.[118] There is no positive evidence that Hardy knew of 'Learning in war-time', but he may have heard of it, for in some ways 'Mathematics in war-time' can be read as a secular mathematical parallel to Lewis's sermon: mathematics should continue, even during war, but it should not be pursued for whatever practical benefits it yields.

Hardy wrote *A Mathematician's Apology* as a more general defence of mathematics, using 'Mathematics in war-time' as the core of the concluding sections. When he asked Cambridge University Press to publish it, he was willing to bear the printing costs himself, but the Syndics (the governing body of the Press) accepted it and decided on an initial print run of 4000.[119]

116 Hooper, 'Introduction', pp. 17–18.
117 Lewis, 'Learning in war-time', p. 50.
118 *Ibid.*, pp. 55–56, 57.
119 Silver, 'In Defense of Pure Mathematics'.

REVIEWS OF THE *APOLOGY*

Alan J. Cain

This essay aims to survey all signed contemporary reviews of the *Apology*.

E. T. Bell, 'Confessions of a Mathematician'
(*The Scientific Monthly*, January 1942)

Bell, mathematician and popular historian of mathematics, described the *Apology* as a 'sardonic confession'. The adjective is better applied to Bell's own short review, where he wrote that the *Apology* was reminiscent of John Henry Newman's *Apologia pro Vita Sua*, a defence of his personal religious opinions, and could be 'specially commended to solemn young men who believe they have a call to preach the higher arithmetic to mathematical infidels'.

Bell saw Hardy's view that 'pure mathematics is on the whole distinctly more useful than applied'[1] as a 'corollary of a classic paradox of G. K. Chesterton'. This may refer to the following of Chesterton's views:

1 Hardy, *Apology*, § 26.

> 'The real trouble with this world of ours is not that it is an unreasonable world, nor even that it is a reasonable one. The commonest kind of trouble is that it is nearly reasonable, but not quite. [...] It looks just a little more mathematical and regular than it is; its exactitude is obvious, but its inexactitude is hidden; its wildness lies in wait.'[2]

Relating to this point, and in response to Hardy's assertion that '"[i]maginary" universes are so much more beautiful than this stupidly constructed "real" one',[3] Bell wrote: 'Well, God, not the mathematical physicist, must take the blame.' He dismissed Hardy's platonism as 'a museum piece from an incredibly credulous past.'

The only positive part of the review is the closing sentence, to the effect that one can disagree with Hardy's conclusions but enjoy his writing style.

Although Bell disparaged the *Apology*, he had a high opinion of Hardy as a mathematician. In 1931, he wrote that only two of the eminent people he had met could be called geniuses: Einstein and Hardy. Each of these men exhibited a 'complete mastery of their stuff' and was an 'absolute master of his trade'.[4]

R. B. Braithwaite, 'Review of *A Mathematician's Apology*'
(*Mind*, October 1941)

The philosopher Braithwaite's review was generally sympathetic, but critical on certain points. He felt that Hardy should have chosen more example theorems from outside number theory.[5] He thought that Euclid's theorem on the infinitude of primes is much more beautiful than the result that $\sqrt{2}$ is irrational, for it is not purely negative: although Hardy gave it as a *reductio*, the proof of the infinitude of primes can be written in a positive form.[6] The

2 Chesterton, *Orthodoxy*, ch. VI.

3 Hardy, *Apology*, § 26.

4 Letter to *The Eagle*, the magazine of Bell's old school, 1931, quoted in Reid, *The Search For E.T. Bell*, p. 255.

5 Braithwaite, 'Review of *A Mathematician's Apology*', p. 420.

6 See Hardy, *Apology*, § 12, annotation 94.

proof of the irrationality of $\sqrt{2}$ is a *reductio* proper, and the theories of proportions and real numbers, which can be traced to it, have, for Braithwaite, 'the æsthetic qualities which it itself lacks'.[7] He also gave what he considered to be immediate counterexamples to Hardy's thesis that it is the dull parts of mathematics that are useful.[8]

Further, Braithwaite criticized Hardy's claim that 'Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not',[9] on the grounds that all thought, including mathematical thought, is dependent on a means of using symbols, and so 'mathematics will vanish with the rest of our intellectual heritage if we revert to our pre-linguistic apehood'.[10]

C. D. Broad, 'Review of *A Mathematician's Apology*' (*Philosophy*, July 1941)

The philosopher of science and historian of philosophy Broad, along with C. P. Snow, read and commented on the manuscript of the *Apology*.[11] Broad's review was generally positive but criticized Hardy on individual points. He agreed with Hardy that mathematics is a form of artistic creation[12] and that its value derives from the discovery of patterns that are beautiful and significant. While agreeing with Hardy that '[h]e who demands some extrinsic justification for it betrays himself as a philistine',[13] Broad thought that Hardy's irritation with such philistinism led him to exaggerate his case when he claimed that it is the 'dull and elementary'[14] parts of mathematics, pure and applied, that work for good or ill:

7 Braithwaite, 'Review of *A Mathematician's Apology*', p. 420.
8 See Hardy, *Apology*, § 25.
9 Hardy, *Apology*, § 8.
10 Braithwaite, 'Review of *A Mathematician's Apology*', p. 421.
11 See Hardy, *Apology*, preface.
12 Broad, 'Review of *A Mathematician's Apology*', p. 325.
13 *Loc. cit.*
14 Hardy, *Apology*, § 25.

> ‘Surely it would be difficult to deny that Newton’s theory of gravitation, Laplace’s and Hamilton’s reduction of the laws of dynamics to the Principle of Least Action, and Maxwell’s theory of the electro-magnetic field are intrinsically beautiful and serious bits of mathematics. And surely they have had extremely important technical applications, both for good and for ill.’[15]

Regarding the motive of ‘immortality’[16] through mathematics, Broad noted that mathematical fame is contingent on circumstances that allow the name of its discoverer to remain associated to a theorem.[17]

Broad also noted[18] that Hardy’s list of mathematicians who died young (Galois, Abel, Ramanujan, Riemann) does not support his thesis that mathematics is ‘a young man’s game’.[19]

Broad agreed that the two examples of serious and beautiful theorems given in §§ 12 and 13 of the *Apology* are ‘quite obviously weighty and beautiful’,[20] but argued that it would have been better if Hardy had distinguished between the theorem and the proof, for Broad felt that in these examples the beauty lies in the reasoning and the seriousness in the result. He did allow that in more complicated examples the theorem and the proof might share both seriousness and beauty.

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15 Broad, ‘Review of *A Mathematician’s Apology*’, p. 326.

16 See Hardy, *Apology*, § 8.

17 ‘No scientific discovery is named after its original discoverer’ is an observation known as ‘Stigler’s law of eponymy’, which was first formulated by Robert K. Merton; see Stigler, ‘Stigler’s law of eponymy’.

18 Broad, ‘Review of *A Mathematician’s Apology*’, p. 324.

19 See Hardy, *Apology*, § 4.

20 Broad, ‘Review of *A Mathematician’s Apology*’, p. 325.

Norman Campbell

(*The News Letter*, 1941)

The annotator has been unable to obtain this review or verify its publication details.

Campbell was a physicist and philosopher of science, and Silver described his review, including a brief quotation:

> 'Norman Campbell took Hardy's assertions at face value. However, if a mathematician's principal motivation is to benefit himself rather than society, he asked "why should we provide … so many more comfortable jobs for mathematicians than for, say, poets or stamp-collectors?"'[21]

I. Bernard Cohen, 'Review of *A Mathematician's Apology* and *Mathematics and the Imagination*'

(*Isis*, June 1942)

Cohen, a historian of science, did not analyse deeply Hardy's arguments. A large part of the review is taken up by the quotation, in full, of the conclusion of the *Apology* (that is, the last two paragraphs of § 29). Cohen suspected that many would disagree with Hardy's views, but all readers would find the book worthwhile.

Ananda Coomaraswamy, 'Review of *A Mathematician's Apology*'

(*The Art Bulletin*, December 1941)

The philosopher and art historian Coomaraswamy engaged primarily with Hardy's discussion of beauty in mathematics. He praised Hardy's analysis of beauty in mathematics, but criticized how Hardy rated mathematical beauty above that found in art. According to Coomaraswamy, Hardy's example of lines from *Richard II* whose outward form is beautiful but whose ideas are false and

21 Silver, 'In Defense of Pure Mathematics'.

trite[22] really proves, not that beauty in art is independent of the validity of the ideas, but that beauty and validity are both relative to context: 'To the Platonist or other traditionalist, and to the reviewer Shakespeare's words are beautiful *and* true, but they are not true for Professor Hardy or in any democratic context'. Coomaraswamy went on to suggest that Hardy should have applied his criteria of intelligibility and economy to art, and his avowed lack of qualifications in aesthetics[23] would have been no hindrance.

Finally, Coomaraswamy agreed that Hardy made the right decision to pursue mathematics, for it was good for his soul, as shown by the kind of monument he would have desired;[24] 'since it is man's first duty to work out his own salvation (from himself), no further defence is needed'.

Arthur Eddington

(*The Cambridge Review*, 21st February 1941)

The annotator has been unable to obtain this review or verify its publication details.

Braithwaite quoted from it Eddington's view on Hardy's choice of the infinitude of the primes and the irrationality of $\sqrt{2}$ as examples of beautiful and serious results: 'One is a perfect gem. The other is an example of mathematics in its most pedestrian mood; its quality is not Art, but a rather pleasant tidiness'.[25] Braithwaite remarked that it is unclear which statement applies to which result.

22 See Hardy, *Apology*, § 10.
23 Hardy, *Apology*, § 11.
24 Hardy, *Apology*, Note.
25 Braithwaite, 'Review of *A Mathematician's Apology*', p. 421.

Félix de Grand'Combe
(*The Journal of Education*, August 1943)

The annotator has been unable to obtain this review or verify its publication details.

'Félix de Grand'Combe' was the pen-name of Félix Boillot, Professor of French at the University of Bristol. Silver twice quoted from his review:

> '"It really is a touching — albeit ostentatious — confession of a local intellectual debility … It is clear that Prof. Hardy is a great mathematician. It is no less clear, from his own showing, that one can be a great mathematician and yet fail to understand things that are readily comprehensible to an ordinary, well-educated mind."
>
> […]
>
> "When Linnaeus devised his wonderful classification of plants he didn't 'make' anything, he merely discovered a pre-existing treasure, explaining and rendering perceptible to all eyes a series of coherent relationships actually present in Nature, but his work altered and clarified our whole conception of the vegetable world; it gave informing reason to apparent chaos, life to what the ancients had seen as a dark welter of 'non-being.'"'[26]

These suggest a very negative tone. In particular, Boillot argued, contra Hardy,[27] that observation, classification, exposition, and appreciation can be just as valuable as discovery.

Graham Greene, 'The Austere Art'
(*The Spectator*, 20th December 1940)

Greene, a novelist, praised highly the *Apology*: 'I know no writing — except perhaps Henry James's introductory essays — which conveys so clearly and with such an absence of fuss the excitement of the creative artist.' He did not criticize Hardy's analysis of

26 Silver, 'In Defense of Pure Mathematics'.
27 Hardy, *Apology*, § 1.

beauty, nor his appreciation of mathematics pursued on purely aesthetic grounds. Graham suggested that the lay reader would be left saddened by their inability to explore the beauties of mathematics the way an expert can.

Desmond McCarthy
(*The Sunday Times*, date unknown)

The annotator has been unable to obtain this review or verify its publication details.

It was mentioned (as being a review of '*An Apology for Mathematics*') by the economist A. C. Pigou in a discussion of how books written by an expert for a general audience can be better evaluated by a reviewer with a good general education.[28] McCarthy was indeed a literary critic for *The Sunday Times* at this time. Pigou's description strongly suggests that McCarthy reviewed the *Apology* positively.

Virginia Modesitt, 'Review of *A Mathematician's Apology*'
(*National Mathematics Magazine*, March 1942)

Modesitt, a mathematician, held the *Apology* to be appealing to any thoughtful person, and to be a challenge to readers to scrutinize the justification for their own lives as closely and frankly as Hardy did. She pointed out that Hardy's discussion of ambition focused those with ability in some field, though her statement that Hardy 'dismisses the problems of the ordinary man as unimportant' is rather stronger than what Hardy said in the *Apology*.[29] She said that Hardy had written in 'a perfect essay form' with 'a very simple, direct, and pleasing style'.

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28 Pigou, 'Newspaper Reviewers, Economics and Mathematics', p. 277.
29 Hardy, *Apology*, § 5.

M. F. A. Montagu, 'Review of *A Mathematician's Apology*'
(*Isis*, Summer 1942)

Montagu, an anthropologist, wrote this one-paragraph review in the 'Sixty-second Critical Bibliography of the History and Philosophy of Science and of the History of Civilization' edited by George Sarton and published in *Isis*. It characterizes the *Apology* as a 'welcome little book', full of 'startling insights', that explains Hardy's 'uselessness'. This seems to be the earliest appearance of the claim that Hardy took pride in the uselessness of his work (see the discussion in the annotator's essay 'Legacy of the *Apology*', pp. 110 sqq.).

—

E. H. Neville, 'Review of *A Mathematician's Apology*'
(*The Mathematical Gazette*, May 1941)

The mathematician Neville accepted Hardy's argument that the justification for devoting oneself mathematics is the intrinsic aesthetic value of the mathematics itself. 'Every mathematician believes this in his heart', he wrote, and complimented Hardy's explanation and defence of this view. However, Neville pointed out a contradiction lurking in the *Apology*: although Hardy stated that most people can appreciate mathematics to some degree, and that the best mathematics gives pleasure after millennia, he nevertheless wrote that '[m]athematics is not a contemplative but a creative subject'.[30] This suggested that Hardy could gain no satisfaction from reading mathematics produced by others, unless it led him to new mathematics of his own. Without it diminishing his respect for Hardy, Neville did not believe this.

—

30 Hardy, *Apology*, § 28.

J. F. Randolph, 'Review of *A Mathematician's Apology*'
(*The American Mathematical Monthly*, June 1942)

Randolph, a mathematician, did not deeply engage with Hardy's arguments. He took a very positive view of the *Apology*, noting that while a mathematician may themself need no apology for mathematics, they can still be grateful to an eminent mathematician for defending it so eloquently.

Frederick Soddy, 'Qui s'accuse s'acquitte'
(*Nature*, January 1941)

The radiochemist Soddy is mentioned in the *Apology* as an example of how people distinguished in their own fields can gain great pleasure in discovering a mathematical result, in his case regarding the 'Hexlet'.[31] Soddy's caustic review begins 'This is a slight book. From such cloistral clowning the world sickens.'[32] The tone throughout is one of biting sarcasm. He disparaged Hardy's distinction of 'real' mathematics and the idea that 'trivial' mathematics is 'ugly in some sort of direct ratio to its usefulness'.[33] As for the assertion that '"[i]maginary" universes are so much more beautiful than this stupidly constructed "real" one',[34] Soddy's response was:

> 'Most scientists [...] believe that the saner outlook on Nature inaugurated by the experimental sciences does reveal the real universe, and that it is not stupidly constructed. Those of them duped by mathematical fantasy are to be put up with rather than pukkha'd.'[35]

A 'real' mathematician, as portrayed by Hardy, was, for Soddy, 'a religious maniac';[36] the field of study belongs in a place of worship,

31 See Hardy, *Apology*, § 10 and the annotations to that section.
32 Soddy, 'Qui s'accuse s'acquitte', p. 3.
33 *Loc. cit.*
34 Hardy, *Apology*, § 26.
35 Soddy, 'Qui s'accuse s'acquitte', p. 3.
36 *Ibid.*, p. 4.

not a university. Hardy's views of education seem to tend toward the religious, and Soddy held that this sort of education is at the root of 'the whole tragedy'[37] (which may refer to World War II or to war in general). Note that, for Soddy, religion included at least some political movements, such as Marxism.[38] Soddy also took a very dim view of the defence of 'real' mathematics as harmless.[39]

Soddy clearly had some appreciation for mathematics, and disparaged the excessive rigour that mathematicians impose on their students, to the detriment of inventiveness.

The review has a curious postscript where Soddy illustrated what he considered 'real' mathematics. He related that once, during a dull meeting, he passed a note to Hardy asking for 'the sum of all the reciprocals of all the odd integers, except unity, raised to the power of each of all the even integers'[40] (from the solution, Soddy must have meant the positive integers). Hardy '*in an incredibly short space of time*'[41] passed back the following answer:

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(2n+1)^{-2m} &= \sum_{n=1}^{\infty}\frac{(2n+1)^{-2}}{1-(2n+1)^{-2}}\\ &= \sum_{n=1}^{\infty}\frac{1}{(2n+1)^2-1}\\ &= \sum_{n=1}^{\infty}\frac{1}{4n(n+1)}\\ &= \frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)=\frac{1}{4}.\end{aligned}$$

Soddy admitted that the question is not difficult, so it is not the mathematics that impressed him, but the speed of Hardy's solution. Real mathematics, for Soddy, thus seemed to involve agility

37 Soddy, 'Qui s'accuse s'acquitte', p. 4.
38 *Loc. cit.*
39 *Loc. cit.*
40 *Ibid.*, p. 5.
41 *Loc. cit.*; emphasis in original.

in problem solving, rather than being something intrinsic to the mathematics itself.

[R. J. L. Kingsford, the general manager of Cambridge University Press, sent a copy of this review to Hardy, writing that 'Soddy's amazing review in *Nature* is a most valuable advertisement'.[42]]

Krishnasami Venkataraman, 'Review of *A Mathematician's Apology*'
(*Current Science*, November 1941)

In his very short review, the chemist Venkataraman's called the *Apology* 'stimulating' but suggested there was no need for such an apology, pointing out the non-trivial role of mathematics in the design of everyday electrical appliances; this seems to ignore Hardy's distinction of 'real mathematics'.

42 Quoted in Silver, 'In Defense of Pure Mathematics'.

LEGACY OF THE *APOLOGY*

Alan J. Cain

A Mathematician's Apology has had a lasting influence: it is cited in discussions of ontology, epistemology, aesthetics, motivations, and creativity. One reason for this is Hardy's beautiful, limpid, supremely quotable prose style; another is the *Apology*'s conciseness; another is the passion that is rare in philosophy of mathematics, at least in writing. Many later discussions were inspired by it, and it compares well with them: as the writer David Foster Wallace put it in 2000, the *Apology*

> 'is the unacknowledged father of most of the last decade's math-prose. There is very little that any of the recent books do that Hardy's terse and beautiful *Apology* did not do first, and with rather less fuss.'[1]

The views Hardy presented in the *Apology*, and in particular the lack of interest in applications, appear to have long held strong appeal for pure mathematicians. Michael Harris related how reading the book as a teenager shaped his view of mathematics long

1 Wallace, 'Rhetoric and the Math Melodrama', p. 2267, note 1.

afterwards.[2] Marcus du Sautoy wrote when he was younger he 'was under the spell of G. H. Hardy's *A Mathematician's Apology*'.[3]

Even when not specifically linked to Hardy, it seems likely that the *Apology*'s defence of mathematics as an aesthetic pursuit was a major cause of the continuing support for this view amongst mathematicians. Recent Hardian echoes can be heard in statements such as: '[a]pplicability is not the reason we work, and plenty that is not applicable contributes to the beauty and magnificence of our subject'[4] and '[a]pplication is not the point [...] Beautiful intellectualizing, that is the satisfaction.'[5]

This essay briefly surveys some important responses and reactions to the *Apology*.

Uselessness and Mischaracterization

Hardy *did not* hold that beautiful mathematics must be useless, and he *did not* aim for or advocate uselessness. These points must be emphasized, for, as discussed below, they form the single most misunderstood aspect of the *Apology*.

The parts of mathematics that Hardy considered useful are given in § 26. Pure mathematics is in some sense more useful than applied since it is the vehicle through which mathematical technique is taught. In terms of the *content* of mathematics, the

> 'general conclusion must be that such mathematics is useful as is wanted by a superior engineer or a moderate physicist; and that is roughly the same thing as to say, such mathematics as has no particular aesthetic merit.'[6]

That is, for Hardy, useful mathematics has 'no particular aesthetic merit'. But Hardy did not argue, here or elsewhere in the *Apology*, that this is a *necessary* truth: the claim is that useful mathematics

2 Harris, *Mathematics without Apologies*, ch. 10.

3 Du Sautoy, *Finding Moonshine*, ch. 10.

4 Rowlett, 'The unplanned impact of mathematics', p. 166.

5 Quoted in Roberts, *Genius At Play*, ch. 16.

6 Hardy, *Apology*, § 26.

happens to have little aesthetic value. It does not follow that useful mathematics *must have* little aesthetic value, or, equivalently, that beautiful mathematics *must be* useless. Hardy simply saw it as a *contingent* fact that 'real mathematics' was useless.[7] He made this contingency explicit: although it is 'the dull and elementary parts' of mathematics that are useful, '[t]ime may change all this'.[8] The contingency is also implicit in his remark that it was only 'very unlikely',[9] not impossible, that a use in war would be discovered for the theory of numbers or relativity. Further, Hardy did not object to applications happening to emerge for mathematics that had aesthetic value:

> 'If the theory of numbers could be employed for any practical and obviously honourable purpose, [...] neither Gauss nor any other mathematician would have been so foolish as to decry or regret such applications.'[10]

Since a loss of beauty could reasonably be thought to cause regret, this implies that if an application arose for some piece of beautiful mathematics, it would not decrease its beauty. Therefore, for Hardy, beauty did not necessarily imply uselessness.

Thus, although Hardy thought that applications in war for number theory and other parts of 'real mathematics' would be unlikely to be found for many years,[11] his error here was only to be unduly optimistic (as a pacifist) about the probability of and time-scale for the emergence of such applications. His incorrectness on this point has no bearing on the views he expressed on aesthetic value.

Hardy explicitly rejected the notion that the uselessness of real mathematics, its lack of 'practical' applications, is a point of pride among mathematicians.[12] The comment that is quoted against him in this regard, that '[a] science is said to be useful if its development

7 Hardy, *Apology*, § 26.
8 Hardy, *Apology*, § 25.
9 Hardy, *Apology*, § 28.
10 Hardy, *Apology*, § 21.
11 Hardy, *Apology*, § 28.
12 Hardy, *Apology*, § 21; Hardy, 'Mathematics in war-time'.

tends to accentuate the existing inequalities in the distribution of wealth, or more directly promotes the destruction of human life'[13] (which was 'a conscious rhetorical flourish'[14]), was written during the First World War. It should be read as a pacifist's sarcastic remark on what *society* considered useful rather than a statement of what *Hardy* himself considered useful. Hardy's own view is given in the *Apology*:

> 'A science or an art may be said to be "useful" if its development increases, even indirectly, the material well-being and comfort of men, if it promotes happiness, using that word in a crude and commonplace way.'[15]

As the quotation from § 21 above makes clear, no mathematician would oppose or regret the application of mathematics for good, but utility was never, and should never, be sought; the goal should be beauty and depth. Hardy did not advocate only the study of useless mathematics: for him, it was simply a fact that real mathematics had no applications in war,[16] and his own work in particular lacked *any* applications.[17] He did not celebrate these facts: he simply drew comfort in inferring that real mathematics could not, at least, be used for evil.[18]

Stressing these points is necessary because part of the legacy of the *Apology* is the conventional 'knowledge' that Hardy made uselessness a requirement for beauty, or even equated uselessness and beauty, or was proud of the uselessness of pure mathematics, or advocated the pursuit of useless mathematics:

1) 'Hardy argued that the utility of a given piece of mathematics is inversely related to its beauty (so that, say, the multiplication table — perhaps the most "useful" part of "mathematics" — is

13 Hardy, 'Prime Numbers', p. 350.
14 Hardy, *Apology*, § 21, note * ; Hardy, 'Mathematics in war-time'.
15 Hardy, *Apology*, § 19.
16 Hardy, *Apology*, § 28.
17 Hardy, *Apology*, § 29.
18 Hardy, *Apology*, § 21.

so devoid of beauty as hardly to deserve the name "mathematics").'[19]

2) 'Hardy went beyond the claim that mathematics is beautiful to also insist that applications of mathematical ideas to the physical world demean those ideas […] — its usefulness detracts from its beauty.'[20]

3) 'Hardy […] argued that there is no place for "ugly" mathematics; important mathematics should always be beautiful. As a consequence, ugly mathematics is applied mathematics, useful mathematics.'[21]

4) 'Mr. Hardy takes great pride in being a "useless mathematician."'[22]

5) Both participants in the following dialogue cheerfully accept the misapprehension:

> '*MP*: With all the applications of elliptic curves to problems in cryptography, it seems that suddenly number theory has become a branch of applied mathematics.
>
> *Selberg*: It would have given great grief to Hardy!
>
> *MP*: That's right. Hardy would not have liked it at all.
>
> […]
>
> *MP*: […] When he gets off into the part about his being so proud that he's never done anything that could be of use to humanity and so on, I wince a bit.'[23]

6) 'Hardy exults particularly in the uselessness of number theory.'[24]

7) 'what G. H. Hardy boastingly called the most useless of all mathematics'.[25]

19 Netz, 'The Aesthetics of Mathematics', p. 253.
20 Clawson, *Mathematical Mysteries*, p. 213.
21 Landri, 'The Pragmatics of Passion', p. 425.
22 Montagu, 'Review of *A Mathematician's Apology*'.
23 Albers & Alexanderson, *Fascinating Mathematical People*, pp. 265–6.
24 Levinson, 'Coding Theory', p. 249.
25 Steen, *Mathematics Tomorrow*, p. 1.

8) 'G. H. Hardy even boasted that his work could never be used for practical purposes.'[26]

9) 'G. H. Hardy, who was proud that "the great bulk of higher mathematics is useless".'[27]

10) 'G. H. Hardy […] boasted that he had never done anything which was remotely useful — though he would be discomforted to know that bank security codes now use the prime numbers he delighted in.'[28]

11) 'G. H. Hardy who expressed the fervent hope that no result he had proven would ever be applied.'[29]

12) 'he fervently hoped that none of his work would ever be applied to anything'.[30]

13) 'He scorned the application of mathematics to anything at all and once expressed the hope that nothing he had ever discovered would have any practical use.'[31]

14) 'his advocacy of the uselessness of mathematics'.[32]

15) 'the *Apology*'s aestheticism and cult of uselessness'.[33]

16) '"Hardy syndrome" (the less useful the better)'.[34]

17) 'Hardyism is the doctrine that one ought only to pursue useless mathematics. This doctrine is given as a purely personal credo in Hardy's *A Mathematicians's Apology*.'[35]

26 Mumford, 'Foreword', p. x.

27 Goodman, 'Mathematics as natural science', p. 187; citation of the *Apology* omitted.

28 Atiyah, 'Address of the President', p. 106.

29 Harary, 'Conditional connectivity', p. 355; citation of the *Apology* omitted.

30 Gardiner, 'Beauty in Mathematics', p. 80.

31 Wells, *Games and Mathematics*, ch. 13.

32 Grattan-Guinness, 'The interest of G.H. Hardy', p. 419.

33 Harris, *Mathematics without Apologies*, ch. 10.

34 Lucas, 'Growth and New Intuitions', p. 56.

35 Davis & Hersh, *The Mathematical Experience*, p. 96.

[Pringle seems to be the only author to note this misconception;[36] the single instance he cites is but a symptom of a widespread disease.]

Quotations 1 to 3 represent Hardy as having equated ugliness with usefulness. Hardy simply held that mathematics that was useful happened to be ugly. (Quotation 3 also seems to contain a *non sequitur*.)

Quotations 4 to 10 suggest Hardy expressed pride in the uselessness of number theory in general and his work in particular. Hardy expressed no pride; he only took solace from his perception that it was useless and his inference that it thus could not be used for harm.

Contra quotations 11 to 13, Hardy expressed no 'hope', fervent or otherwise, about the enduring uselessness of his work. He simply stated as a fact that he had 'never done anything "useful"' and that, in all likelihood, his work would never make 'the least difference' to the world.[37]

Finally, unlike the claims of quotations 13 to 17 and the suggestions in quotations 5 and 10 that he would have been aggrieved by later applications of number theory, Hardy in no way advocated uselessness. As mentioned above, he drew some comfort from not having caused harm, but he held that mathematics should be pursued for aesthetic reasons, *independently of possible applications*.

[An uncharitable reader, thinking that mathematicians would understand that 'being useful is not a goal' does not imply 'not being useful is a goal', might consider as deliberate calumnies the contents of some of these quotations.]

The effects of this kind of mischaracterization are not limited to discussions of the value of mathematics. The principle in population genetics now known as the Hardy–Weinberg law was published by Hardy in 1908 in a letter to *Science*;[38] Crow suggested the following explanation for Hardy's choice of venue:

36 Pringle, *A Hardian Theory*, p. 5.

37 Hardy, *Apology*, § 29.

38 Hardy, 'Mendelian proportions in a mixed population'.

> 'It must have embarrassed him that his mathematically most trivial paper is not only far and away his most widely known, but has been of such distastefully practical value. He published this paper not in the obvious place, *Nature*, but across the Atlantic in *Science*. Why? It has been said that he didn't want to get embroiled in the bitter argument between the Mendelists and biometricians. I would like to think that he didn't want it to be seen by his mathematician colleagues.'[39]

Yet, given that Hardy did not advocate uselessness, there is little reason to believe Hardy would have found the law 'distastefully practical'. He did find the result trivial (he says as much in the letter: 'the very simple point which I wish to make'[40]). Would this have embarrassed him? This would rely on his *knowing* that this letter became widely known. Punnett, the geneticist who drew Hardy's attention to the problem the letter addressed, knew that Hardy had 'not the slightest interest in genetics' and thus phrased it as a mathematical problem.[41] This tends to count against Hardy having been aware of how famous his result became in genetics.

The Value of Mathematics

H. C. Plummer. The astronomer H. C. Plummer wrote a letter to *The Mathematical Gazette* in December 1941[42] which seems to be the first published reaction to the *Apology* other than a review. He took issue with Hardy's assertion that it was 'very hard to find an instance of a first-rate mathematician who has abandoned mathematics and attained first-rate distinction in any other field'.[43] Plummer pointed out that one can

> 'think of Des Cartes, Pascal, Barrow, Wren, Newton and Leibniz to see that there have been mathematicians, and

39 Crow, 'Eighty Years Ago', p. 474.
40 Hardy, 'Mendelian proportions in a mixed population', p. 49.
41 Punnett, 'Early Days of Genetics', p. 9.
42 Plummer, 'The Mathematician and the Community'.
43 Hardy, *Apology*, § 4.

eminently serious ones too, who have not thought it necessary, and have even thought it wrong, to devote the whole of their lives to the pursuit of mathematics.'[44]

This led Plummer to the question of whether it is 'better' to seek mathematical fame or to contribute to society in some other way. Plummer acknowledged that few people have the mathematical gifts to be confronted with this choice, but he was unconvinced by Hardy's answer. Plummer pointed out that the decision was easy for someone of Hardy's time because of the particular social context: there was an open and clear path through scholarships to fellowships that led into mathematics, and few turnings that led back out.[45]

Plummer concedes that the very greatest mathematicians — those with genius-level talent — probably do serve society best by doing mathematics. But it may not be fair either on an individual or social level that those of lesser but still exceptional talent be channelled into a system that treats them as 'academic fodder to be made into specialized mathematicians'.[46] Plummer concluded that society suffered because of this system.

John von Neumann. Von Neumann's 1947 essay 'The Mathematician' does not cite the *Apology* (nor any other work), but it certainly suggests that von Neumann was aware of Hardy's position. Von Neumann's description of how mathematicians judge their own success is in agreement with Hardy: 'the mathematician's subjective criterion of success, of the worth-whileness of his effort, is very much self-contained and aesthetical and free (or nearly free) of empirical connections.'[47]

Furthermore, the features that von Neumann gave of beautiful theorems and theories are reminiscent of Hardy:

44 Plummer, 'The Mathematician and the Community', p. 301.

45 *Loc. cit.*

46 *Loc. cit.*

47 Von Neumann, 'The Mathematician', pp. 6–7.

> ‘One expects a mathematical theorem or a mathematical theory not only to describe and to classify in a simple and elegant way numerous and a priori disparate special cases. One also expects “elegance” in its “architectural,” structural makeup. Ease in stating the problem, great difficulty in getting hold of it and in all attempts at approaching it, then again some very surprising twist by which the approach, or some part of the approach, becomes easy, etc. Also, if the deductions are lengthy or complicated, there should be some simple general principle involved, which “explains” the complications and detours, reduces the apparent arbitrariness to a few simple guiding motivations, etc.’[48]

Using other language, von Neumann has described the Hardian qualities of unexpectedness (‘some very surprising twist’) and inevitability (‘reduces the apparent arbitrariness’).

However, von Neumann was cautious about mathematics venturing too far away from applications. It is not bad to pursue a purely mathematical topic for its for own sake, *provided that* the field is guided either by its relationship to other areas that have a greater empirical component, or else by ‘the influence of men with an exceptionally well-developed taste’.[49] Without this guidance, there is a ‘danger of degeneration’: that the subject ‘will separate into a multitude of insignificant branches, and […] will become a disorganized mass of details and complexities’.[50] In this situation, the only way for the discipline to recover is via an infusion of new ideas from empirical experience.

Balthasar van der Pol. The physicist van der Pol disagreed with Hardy on the applicability of number theory, and gave a number of actual and potential applications. However, van der Pol is at pains to do so respectfully, partly because Hardy was dead and unable to

48 Von Neumann, ‘The Mathematician’, pp. 8–9.
49 *Ibid.*, p. 9.
50 *Loc. cit.*

reply, and because of the debt he felt toward Hardy for the beautiful mathematics he created.[51] He was clearly very sympathetic to Hardy, describing the *Apology* as a 'charming and sometimes even pathetic essay'.[52]

C. Stanley Ogilvy. Ogilvy, author of *Through the Mathescope*, reads like a less assertive version of Hardy: pure mathematics is done for its own sake; applicability is not a motivation. The pure mathematician

> 'does mathematics for the same reasons that the artist paints: for the fascination of the subject itself; for the satisfaction of producing something that to himself and to his colleagues is beautiful; and, if he has a spark of real genius, because he can't help it. Mathematics, for all its scientific dress, is very like an art.'[53]

Indeed, Ogilvy explicitly compared his discussion with the *Apology*, although he said that Hardy 'sometimes overstates the case'.[54]

However, for Ogilvy, to have *any value*, a mathematical work must be 'serious' in the Hardian sense: it must have connections to established ideas, lest it resemble a work of modern art whose meaning escapes everyone, including the artist.[55]

Gerald Whitrow. The cosmologist and historian of science Gerald Whitrow argued in 1956[56] against Hardy's view that 'Archimedes will be remembered when Aeschylus is forgotten',[57] quoting the classicist Giles Murray for support:

> 'The time has come for Euclid to be superseded; let him go. He has surely held the torch for mankind long enough; and

51 Van der Pol, 'Radio Technology and the Theory of Numbers', p. 477.
52 *Loc. cit.*
53 Ogilvy, *Through the Mathescope*, p. 6.
54 *Ibid.*, p. 131.
55 *Ibid.*, p. 7.
56 Whitrow, 'The Study of the Philosophy of Science', p. 194.
57 Hardy, *Apology*, § 8.

> books of science are born to be superseded. […] But when we read Homer or Aeschylus, if once we have the power to admire and understand their writing, we do not for the most part have any feeling of having got beyond them.'[58]

But Murray's view here does not stand in opposition to Hardy's: Murray's suggestion is to give up Euclid's *Elements* as a textbook, not that the ideas it contains are obsolete, unlike those in an ancient treatise in medicine or mechanics. Few today read Euclid's or Archimedes' works, even in translation, but the mathematics they contain lives on: '*languages die but mathematical ideas do not*'.[59]

Milton Babbitt. In 1958 Babbitt, a composer who was trained in mathematics, wrote the celebrated essay 'Who Cares if You Listen?', which argues that the development of music, like that of mathematics, science, and philosophy, has passed beyond what can be appreciated by the non-specialist.[60] Modern composers would benefit themselves and their music if they did not continue to aim at a public audience, and instead pursued 'a private life of professional achievement';[61] it is for the university to 'provide a home for the "complex," "difficult," and "problematical" in music'.[62]

Babbitt's justification for music parallels Hardy's for mathematics: the creation of something of aesthetic value, even if it is only accessible to a minority. Babbitt had been a professional mathematician, and, though he does not cite it, may have known the *Apology*. However, it is also possible that he reached similar conclusions to Hardy from similar Moorean premises.

Eugene Wigner. Wigner's famous essay on 'The unreasonable effectiveness of mathematics in the natural sciences' may have been

58 Murray, 'The Value of Greece', p. 5.
59 Hardy, *Apology*, § 8.
60 Babbitt, 'Who Cares if You Listen?', p. 40.
61 *Ibid.*, p. 126.
62 *Loc. cit.*

partly inspired by Hardy's *Apology*: although Wigner did not explicitly cite Hardy, the section of the essay entitled 'What is Mathematics?' emphasizes the role of aesthetics in the motivation for mathematics and the choice of direction in which it develops. These aesthetic choices are one of the reasons Wigner found mysterious the applicability of mathematics in science. Wigner's article has been called an 'over-reaction' to mathematicians such as Hardy.[63]

Symposium. In a 1961 symposium on the place of applied mathematics in research and education, George F. Carrier, Richard Courant, Paul C. Rosenbloom, and Chen-Ning Yang discussed matters such as usefulness, aesthetics, and motivation.[64] The *Apology* clearly sat in the background of the discussion, and two of the four speakers explicitly discussed it.

Of the four speakers, Courant was the most obviously opposed to Hardy's views. He dismissed as 'old blasphemous nonsense'[65] the idea of mathematics being ultimately justified on an aesthetic level, or indeed on any other level of purely intellectual satisfaction. This is not to say that he denied that mathematics can be beautiful, but simply that the pursuit of such beauty cannot justify mathematics. He further argued that '[m]athematics must not be allowed to split and to diverge towards a "pure" and an "applied" variety.'[66] That is, contra Hardy, he did not see any division between pure and applied mathematics in terms of their content. Rather, there is a only a difference in motivation between pure and applied *mathematicians*: that latter has 'a profound interest in the connection between mathematics and what may be called reality'. The mathematical platonist Hardy might have questioned this distinction, since he thought that research brought the mathematician, much more than the physicist, into close contact with 'a reality far more intense and far more rigid than the dubious and elusive reality of

63 Goodman, 'Mathematics as natural science', p. 187.
64 Carrier et al., 'Applied mathematics'.
65 *Ibid.*, p. 298.
66 *Loc. cit.*

physics'.[67]

Rosenbloom was more sympathetic to the role of aesthetic value in mathematics and science generally. He noted that there was only 'very inadequate evidence' for the general theory of relativity, but that it was accepted on grounds of elegance, simplicity, economy, and unity. Thus to teach applied mathematics purely from the perspective of usefulness is to elide a principal aspect of how judgements are actually made in applied mathematics.[68]

Rosenbloom also made an incisive observation which may go some way towards explaining some of the disdain in which some mathematicians hold applications. In certain institutions, mathematics departments had to struggle to win for themselves an existence independent of servicing other departments. This may have led to a certain pride in developing mathematics for its own sake, rather than for its applications.[69] Rosenbloom further pointed out that most people who will have to teach applied mathematics became mathematicians for aesthetic reasons, and that therefore 'if you want to get applied mathematics into the curriculum of the college and the graduate school, you are going to have to present it in such a way as to emphasize that applied mathematics can also be beautiful, can also be deep.'[70] Another speaker, Carrier, emphasized that this is indeed possible: 'The use of mathematics in science and elsewhere can be as challenging, as esthetically pleasing, and as valuable to society, as the "pure" self-contained discipline.'[71]

Yang suggested that Hardy's training in pure mathematics is responsible for his lack of appreciation of the beauty found in the application of mathematics to the physical world; for Yang, this appreciation is vital for an applied mathematician.[72] Indeed, Hardy's views of applied mathematics are prescriptive of what mathematics

67 Hardy, 'The theory of numbers', p. 381.
68 Carrier et al., 'Applied mathematics', p. 305.
69 *Ibid.*, p. 307.
70 *Loc. cit.*
71 *Ibid.*, pp. 316–7.
72 *Ibid.*, p. 310.

should *not* be.[73]

C. P. Snow. Hardy's friend C. P. Snow read the *Apology* in manuscript and held it in high regard. He echoed Greene's review:[74] the *Apology* 'is one of the most beautiful statements about the creative mind ever written or ever likely to be written.'[75] Snow was certainly aware of Greene's review, for he referred to it in his biographical essay of Hardy.[76] More generally, he admired Hardy as a creator of art:

> 'he was clearly superior to Einstein or Rutherford or any other great genius [...] at turning any work of the intellect, major or minor or sheer play, into a work of art. It was that gift above all, I think, which made him, almost without realizing it, purvey such intellectual delight.'[77]

Snow also treated as a novelist some of the same issues Hardy considered in the *Apology*. His *Strangers and Brothers* cycle of novels, though less well-known today, still forms a fascinating portrait of mid-twentieth century Britain seen through the eyes of the narrator Lewis Eliot, variously a lawyer, legal academic, and civil servant. The sixth novel in the series, the *The New Men*, published in 1954, is centred on the United Kingdom's atomic weapons research during the Second World War. In a part of the story set in 1943, Snow has Eliot dwell on a scientist's motivations:

> 'What had made him a scientist? How would he justify it?
> [...]
> [...] Science, said Mounteney, had been the one permanent source of happiness in his life; and really the happiness was a private, if you like a selfish, one. It was just the happiness he derived from seeing how nature worked; it would

73 Carrier et al., 'Applied mathematics', p. 309.

74 Greene, 'The Austere Art'; see p. 103.

75 Snow, 'The Classical Mind', p. 812.

76 Snow, 'Foreword', p. 13.

77 *Loc. cit.*

> not have lost its strength if nothing he had done added sixpence to practical human betterment. […]
>
> [… It] was beginning to seem too private, not enough justification for a life. Mounteney would have liked to say, as he might have done in less austere times, that science was good in itself; he felt it so; but in the long run he had to fall back on the justification for himself and other scientists, that their work and science in general did practical good to human lives.'[78]

This is followed by the characters discussing how science, even if held responsible for all deaths in modern warfare, has kept alive a far greater number, but that the atomic bomb might conceivably tip the balance.

It is difficult to imagine that Snow was not conscious of Hardy's *Apology* when he wrote these lines: note in particular the resonance of Snow's 'the one permanent source of happiness in his life' with Hardy's 'one great permanent happiness of my life'.[79]

[Besides the *Apology* itself, there are other, more minor, Hardian — or, at least, Hardy-esque — influences on Snow's novels. Snow discussed *The Masters* with Hardy, who contributed some ideas.[80] Snow admitted that Hardy appeared in his novels, though 'in a form so transmogrified that no one has ever noticed'.[81] Some of the facets of Hardy's personality are incorporated into the character of Adrian Davidson, who appears in *Homecomings*, *The Sleep of Reason*, and *Last Things*.[82] In particular, Davidson uses an expression Snow attributed to Hardy, who exhibited a suspicion of technology: 'If you *fancy yourself* at the telephone'.[83]]

Norman Levinson. Levinson's 1970 article on the theory of error-correcting codes is an exposition of an area of mathematics that is

78 Snow, *The New Men*, ch. 13.
79 Hardy, *Apology*, § 29.
80 Snow, 'Foreword', p. 49.
81 Snow, *Variety of Men*, p. xii.
82 Tredell, *C.P. Snow*, p. 13.
83 Snow, *Homecomings*, ch. 37; cf. Snow, 'Foreword', p. 48.

both beautiful and useful. It is marred, however, because its stated aim is to refute a view of Hardy that includes the mischaracterization that Hardy was proud of the uselessness of his work (see pp. 110 sqq.).[84] It *does* successfully refute the idea that Hardy's 'real mathematics' lacks application. Although the theories of relativity and quantum mechanics had clearly become useful in the time since Hardy wrote and included them in 'real mathematics', Levinson made the observation that Hardy was not an expert these fields. Thus a more secure refutation of this idea would use an example from pure mathematics. Levinson agreed with the aesthetic qualities Hardy identified in beautiful theorems and proofs and his aim is thus to show that these qualities are exhibited by the role of finite fields in coding theory; he also notes that the quadratic reciprocity theorem, which Hardy thought beautiful,[85] enters into coding theory.[86] While many works *mention* the applicability of Hardy's 'real mathematics' (in, for example, cryptography), Levinson's article appears to be the only work that actually makes the case at length. Note, however, that the theory of error-correcting codes only began to develop after Hardy's death.

Louis J. Mordell. Mordell, who succeeded Hardy as Sadleirian Professor of Pure Mathematics at Cambridge, wrote in 1970 a critique of the *Apology*, including the biographical essay by Snow in the 1967 edition. Mordell made the important observation that Hardy often stated his views in absolute terms, without allowing for exceptions or limitations.[87] In opposition to the first lines of the *Apology*, Mordell noted occasions when Hardy spoke 'about' mathematics.[88] To these, one could add examples such as Hardy's lectures on the notion of proof,[89] on number theory,[90] or against

84 Levinson, 'Coding Theory', p. 249.
85 Hardy, *Apology*, § 12.
86 Levinson, 'Coding Theory', p. 250.
87 Mordell, 'Hardy's "A Mathematician's Apology"', p. 834.
88 *Ibid.*, pp. 831–2.
89 Hardy, 'Mathematical proof'.
90 Hardy, 'The theory of numbers'.

the Mathematical Tripos.[91]

Mordell pointed out that a mathematician cannot always be focussed on producing new results.[92] Now, according to Snow, Hardy said that four hours per day is the limit for a mathematician doing creative work.[93] But Mordell here meant extended 'fallow periods' in which the mathematician may contribute to the advancement of mathematics in other ways, such as through exposition or administrative work.[94]

To the Hardian triad of purely aesthetic qualities in results and proofs, namely unexpectedness, inevitability, and economy, Mordell added

> 'simplicity of enunciation. The meaning of the result and its significance should be grasped immediately by the reader, and these in themselves may make one think, what a pretty result this is. It is, however, the proof which counts. This should preferably be short, involve little detail and a minimum of calculations. It leaves the reader impressed with a sense of elegance and wondering how it is possible that so much can be done with so little.'[95]

The last two sentences seem to overlap with economy, but simplicity of enunciation clearly encompasses the concepts and notation used to express the result and proof; these are clearly distinct from Hardy's qualities.

Mordell also considered what would count as 'ugly mathematics', which Hardy mentioned in the *Apology* but does not define beyond describing of ballistics and aerodynamics as 'repulsively ugly':[96] ugly theorems and proofs include

> 'those involving considerable calculations to produce results of no particular interest or importance; those involving such

91 Hardy, 'The Case Against the Mathematical Tripos'.
92 Mordell, 'Hardy's "A Mathematician's Apology"', p. 834.
93 Snow, 'Foreword', p. 32.
94 Mordell, 'Hardy's "A Mathematician's Apology"', p. 834.
95 *Loc. cit.*
96 Hardy, *Apology*, § 28.

> a multiplicity of variables, constants, and indices, upper, lower, right, and left, making it very difficult to gather the import of the result; and undue generalization apparently for its own sake and producing results with little novelty. I might also mention work which places a heavy burden on the reader in the way of comprehension and verification unless the results are of great importance.'[97]

Mordell gave several counterexamples to the uselessness of 'real mathematics': including the application of conic sections to the orbits of the planets and of Riemannian geometry in relativity.[98] He also noted that many new disciplines such as game theory and communications theory make increasing use of pure mathematics.[99]

Finally, Mordell took issue with the view that mathematics is not a contemplative subject:

> 'Many people can derive a great deal of pleasure from the contemplation of mathematics, e.g., from the beauty of its proofs, the importance of its results, and the history of its development. But alas, apparently not Hardy.'[100]

However, what Hardy wrote was:

> 'Mathematics is not a contemplative but a creative subject; no one can draw much consolation from it when he has lost the power or the desire to create.'[101]

This statement is perhaps more nuanced than the position Mordell inferred: it suggest that Hardy could no longer enjoy the contemplation of mathematics *after* having lost his creative ability; contemplation would be a reminder of the passing of creative ability, leaving Hardy unable to enjoy it.

97 Mordell, 'Hardy's "A Mathematician's Apology"', p. 835.
98 *Loc. cit.*
99 *Ibid.*, pp. 835–836.
100 *Ibid.*, p. 834.
101 Hardy, *Apology*, § 28.

Paul Halmos. Halmos's provocatively-titled 1981 essay 'Applied Mathematics Is Bad Mathematics' does not explicitly cite the *Apology* (or indeed any other work), but it clearly draws on Hardy's thought both in the general framing of its argument and in certain of its features (for example, using, chess as an example of trivial mathematics[102]). Halmos argues that from one perspective, the difference between pure and applied mathematics is perhaps no better defined than the difference between pure and applied literature: there is a continuous spectrum ranging from one to another. Furthermore, aesthetics cannot serve as a distinction: applied mathematics can be beautiful too.[103]

There is a difference in motivations and attitudes between pure and applied mathematics:

> 'The motivation of the applied mathematician is to understand the world and perhaps to change it; the requisite attitude (or, in any event, a customary one) is one of sharp focus [...]. The motivation of the pure mathematician is frequently just curiosity; the attitude is more that of a wide-angle lens than a telescopic one'.[104]

Such fundamental differences are probable causes of the more different traditions in standards of exposition, aesthetics and 'perhaps even logical rigor'.[105] In particular, a pure mathematician will award another's work the highest praise by calling it 'beautiful'; an applied mathematician might prefer 'ingenious' or 'powerful'.[106]

Halmos ultimately explains his title with the aid of an artistic analogy:

> 'A portrait by Picasso is regarded as beautiful by some, and a police photograph of a wanted criminal can be useful, but

102 Halmos, 'Applied Mathematics Is Bad Mathematics', p. 17; cf. Hardy, *Apology*, §§ 10–11.

103 Halmos, 'Applied Mathematics Is Bad Mathematics', pp. 12–13.

104 *Ibid.*, p. 14.

105 *Loc. cit.*

106 *Ibid.*, p. 15.

> the chances are that the Picasso is not a good likeness and the police photograph is not very inspiring to look at. Is it completely unfair to say that the portrait is a bad copy of nature and the photograph is bad art?'[107]

Halmos's point is that although the best discoveries of applied mathematics are great *applied mathematics* (mathematics as a mirror of nature) and deserve the highest respect and praise *as applied mathematics*, they are nevertheless bad *mathematics* (mathematics as an art).[108] Thus in the end Halmos's 'mathematics' is implicitly very close to Hardian 'real mathematics'.

Nicholas Young. In the afterword to his 1988 textbook on Hilbert space, Young engaged with Hardy's arguments regarding the value of mathematics.[109] He made the important point that Hardy justifies *individuals* doing mathematics: the book is *A Mathematician's Apology*, not *An Apology for Mathematics* (although Hardy does describe his aim in those terms[110]). Young suggested instead that mathematicians must justify themselves to society:

> 'This does not mean we must be crudely utilitarian, but it does mean we should be ready to give an account of the part played by mathematics as a whole in science, technology, industry, commerce and government, and that we should be subject to political judgement as to the resources it deserves.'[111]

He did not deny that the aesthetic value of mathematics exceeds the value of the other fields it supports; he held that the argument for supporting mathematics research should be part of the argument for science and technology as a whole.[112] Thus his fundamental

107 Halmos, 'Applied Mathematics Is Bad Mathematics', p. 20.

108 Halmos, 'Applied Mathematics Is Bad Mathematics', p. 20; see also Albers & Alexanderson, *Mathematical People*, p. 127.

109 Young, *An introduction to Hilbert space*, Afterword.

110 Hardy, *Apology*, § 2.

111 Young, *An introduction to Hilbert space*, p. 230.

112 *Ibid.*, p. 231.

objection to the *Apology* is that it 'overemphasizes the individual':[113] a mathematician does not do his work alone but within a social structure. Science as a process is located in the activities of those who do it; '[i]t is a grand structure of which no one person can see more than a tiny part.'[114]

Young also asserted that 'there is much less agreement among mathematicians than Hardy implies as to what is boring, what is beautiful and what is "real mathematics"',[115] although he does not cite evidence. He also rejected the idea that the applicable parts of mathematics are dull.

David Henley. In 1995 Henley pointed out, apropos of Hardy's analysis of seriousness, that

> 'is possible that the relationship of interest or significance between mathematical ideas is, as Hardy implies when he says it is revealed by proof,[116] *syntactic*, requiring discussion at the meta level, not the object level. And the interest of a mathematical entity may thus depend partly on abstract properties, not of the entity, but of the formulae which define it, i.e. they are properties of mathematical language rather than of the mathematical entities to which it refers.'[117]

This does not contradict Hardy's avowed platonism, for it says nothing about the ontology of mathematical objects. But it emphasizes two important points that Hardy does not: the value of mathematics does not *reside in* the mathematical world; and the value of mathematics may depend on how we represent it.

Gian-Carlo Rota. Rota, a mathematician and philosopher, fundamentally disagreed with the idea that there is beauty in mathematics; Rota instead reinterpreted 'mathematical beauty' as 'enlightenment'. Specifically, contra Hardy, he says that

113 Young, *An introduction to Hilbert space*, p. 231.
114 *Loc. cit.*
115 *Loc. cit.*
116 Hardy, *Apology*, § 15.
117 Henley, 'Syntax-directed discovery', p. 247.

> ‘one can find instances of very surprising results which no one has ever thought of classifying as beautiful. For example, Morley’s theorem […] is unquestionably surprising, but neither the statement nor any of the proofs of the theorem can be viewed as beautiful […]. A great many theorems of mathematics, when first published, appear to be surprising; thus for example some twenty years ago the proof of the existence of non-equivalent differentiable structures on spheres of high dimension was thought to be surprising, but it did not occur to anyone to call such a fact beautiful, then or now.’[118]

However, Rota made an unhappy choice of examples: Oakley & Baker[119] and Bankoff[120] found Morley’s theorem beautiful, while Monastyrsky seemed to contradict Rota’s other example, describing the

> ‘beautiful construction of the different differential structures on the seven-dimensional sphere. […] The original proof of Milnor was not very constructive but later E. Briscorn [*sic*[121]] showed that these differential structures can be described in an extremely explicit and beautiful form.’[122]

Adam Pringle. Pringle’s 2006 essay aimed to develop a philosophical theory of mathematical beauty based upon Hardy’s *Apology*.[123] As Pringle pointed out, Hardy did not aim to present a systematized philosophical theory of mathematical beauty. Pringle, however, viewed the account of mathematical beauty in the *Apology* as the natural starting-point to develop such a theory, for Hardy at least provided foundations to build upon. As Pringle pointed out, and as

118 Rota, ‘Phenomenology of Mathematical Beauty’, p. 172.
119 Oakley & Baker, ‘The Morley trisector theorem’, p. 738.
120 Bankoff, ‘The Beauty and the Truth of the Morley Theorem’, p. 294.
121 Egbert Brieskorn.
122 Monastyrsky, ‘Some trends’, p. 4.
123 Pringle, *A Hardian Theory*, Abstract.

evidenced by this essay, the *Apology* has become the 'locus for discussions of mathematical beauty.'[124] Whether Hardy is the first to 'even come close to presenting a theory of mathematical beauty'[125] is dubious; Francis Hutcheson's discussion, originally published in 1725, must surely qualify.[126]

Pringle proceeded to make a careful analysis of the kinds of value found in results and proofs, starting from Hardy's text but making a more precise analysis. In particular, there is a more exact study of how beauty and seriousness interact, a point on which Hardy seemed less than clear. This careful analysis also leads Pringle to be apparently the only author to note the mischaracterization discussed on pp. 110 sqq..

Marc Lange. Lange suggested that Hardy's praise of proofs that do not involve consideration of cases is implicitly connected to the notion of 'coincidence' in mathematics.[127] Lange initially characterized non-coincidental results in terms of their proofs as follows:

> 'Suppose we take that single component of the non-coincidence and make each step of the proof as logically weak as it can afford to be while still allowing the proof to explain that component. Then the weakened proof remains able to explain each of the non-coincidence's other components as well'[128]

Although Lange refined this characterization, the connection with case analysis is clear: a case-by-case treatment does not satisfy this criterion; fundamentally, such a proof cannot demonstrate that the result is not a coincidence. As Lange pointed out elsewhere, a proof by division into cases 'fails to identify the real reason that the theorem holds, whereas a unifying proof supplies this explanation.'[129] This connects to Rota's reinterpretation of mathematical

124 Pringle, *A Hardian Theory*, Abstract.
125 *Ibid.*, Abstract.
126 Hutcheson, *Inquiry Concerning Beauty*, § III.
127 Lange, 'What Are Mathematical Coincidences?', pp. 337–8.
128 *Ibid.*, p. 321.
129 Lange, 'Depth and explanation in mathematics', p. 197.

beauty in terms of enlightenment, for case-by-case proofs, which seem (*ceteris paribus*) to be generally acknowledged as ugly, are not enlightening.

Michael Harris. Harris's 2015 book *Mathematics without Apologies* is, as its title suggests, in part a self-conscious response to the *Apology*. Harris's goal was not to offer a justification for the pursuit of mathematics (hence 'without apologies'), but actually to give 'a sense of the mathematical life'.[130] In particular, Harris aimed to portray the social aspects of mathematics, considering aspects of how leaders emerge, how they influence the direction of mathematical development, the process by which mathematical theories are accepted and reshape the field, how institutional structures shape research. By contrast, Hardy's account, while mentioning recognition by one's peers as a motivation for mathematics,[131] is nevertheless fundamentally an individualistic account.

In a final chapter, Harris placed the *Apology* in its context, noting the influence of G. E. Moore and the Bloomsbury Group. He also recorded the influence that Hardy's attitudes (or, more precisely, his perceived attitudes; see pp. 110 sqq.) have with professional mathematicians, including Harris himself, to the extent that they have become near-unquestioned assumptions.

Adam Rieger. Rieger's 2018 article points out that Hardy's platonism illuminates an important distinction: 'In mathematics, the main aesthetic value lies with the thing represented, not the representation.'[132] Of course, an individual presentation of a piece of mathematics may be described in aesthetic terms,[133] but beauty is mainly seen in theorems or proofs. Here mathematics differs from literature, painting, and sculpture, where the beauty is seen in

130 Harris, *Mathematics without Apologies*, Preface.

131 Hardy, *Apology*, §§ 7–8.

132 Rieger, 'The Beautiful Art of Mathematics', p. 14.

133 Cf. 'performance' in science, discussed in Hofstadter, *Le Ton Beau de Marot*, pp. 363–4.

the representation. However, Rieger did not think there is a sharp contrast:

> '[a]rguably the most valued paintings have beautiful subjects, as well as being themselves beautiful representations; part of the what the artist is commended for is having successfully conveyed a beautiful part of reality.'[134]

In opposition to Rieger, one could note that some of the supreme works of literature depict narratives, characters, or settings that are certainly not beautiful: Dante's *Inferno*, say, or much of tragic literature.

'A young man's game'

The oft-cited passage from the *Apology* that 'mathematics, more than any other art or science, is a young man's game'[135] is contentious. Clearly, Hardy himself felt his own powers had declined: as C. P. Snow noted in his biographical essay of Hardy, the *Apology* is

> 'a book of haunting sadness. [...] a passionate lament for creative powers that used to be and that will never come again.'[136]

But the broader idea that mathematicians do their best work when young has been debated. As C. D. Broad noted in his review,[137] Hardy's list of mathematicians who died young does not give any support to this thesis. However, Hermann Weyl stated that he agreed wholeheartedly with Hardy on this point.[138]

The psychiatrist Anthony Storr seemed to be responding to Hardy when he suggested a hypothesis for an underlying cause for Hardy's thesis:

134 Rieger, 'The Beautiful Art of Mathematics', p. 15.
135 Hardy, *Apology*, § 4.
136 Snow, 'Foreword', pp. 50–1.
137 Broad, 'Review of *A Mathematician's Apology*', p. 324; see pp. 99 sq..
138 Weyl, 'Axiomatic Versus Constructive Procedures in Mathematics'.

> ‘there is some reason to suppose that people who become scientists are temperamentally governed by the notion of emotional self-control, whilst those who turn towards the arts are governed by the notion of self-expression […]. These temperamental differences may also be related to the fact that, whereas most mathematicians and physical scientists produce their best work early in life, artists tend to reach their peak later.’[139]

(Storr was certainly aware of the *Apology*, for he cited it in another context in the same article.[140])

Giving examples of mathematicians who produced good work in later life, contra Hardy, has almost become a cliché. It is perhaps an irony that Hardy’s longtime collaborator J. E. Littlewood is a counterexample to Hardy’s thesis; Littlewood did important work in his seventies and his last paper appeared when he was 87.[141] Indeed, Littlewood connected long life with continuing mathematical work:

> ‘Mathematics is very hard work, and dons tend to be above average in health and vigor. Below a certain threshold a man cracks up; but above it, hard mental work makes for health and vigor (also — on much historical evidence throughout the ages — for longevity).’[142]

Hardy’s successor Mordell retired from the Sadleirian chair in 1953 at the age of 65; more than half of his publications appeared after his retirement.[143] Mordell himself seemed to accept a nuanced version of Hardy’s diagnosis:

139 Storr, ‘Bridging the Two Cultures’, p. 72.

140 *Ibid.*, p. 73.

141 Hersh & John-Steiner, *Loving + Hating Mathematics*, p. 252.

142 Littlewood, *Miscellany*, p. 195.

143 `http://www-history.mcs.st-and.ac.uk/Biographies/Mordell.html`; Mordell’s retirement age is given incorrectly in Hersh & John-Steiner, *Loving + Hating Mathematics*, p. 252.

> 'We all know only too well that with advancing age we are no longer in our prime, and that our powers are dimmed and are not what they once were. Most of us, but not Hardy, accept the inevitable.'[144]

Yet Mordell also noted Sydney Chapman[145] and himself as examples of '[g]reat activity among octogenarians'.[146]

Hersh & John-Steiner, in response to Hardy, gave a number of examples of opinions about aging among mathematicians, and examples of mathematicians who did their best work late in life; they also presented the results of a survey on age and mathematics.[147]

The Two Cultures

Du Sautoy assigned to the opening sentences of the *Apology* some of the blame for the existence of the two cultures: their insistence that the proper function of the mathematician is creation rather than exposition has become so embedded in mathematical culture that pure mathematics has become very insular.[148] This is possible, although it must be tempered in a number of respects. First, the two cultures divide was in evidence before the *Apology* appeared, as Hardy himself observed to Snow:

> 'I remember G. H. Hardy once remarking to me in mild puzzlement, some time in the 1930's: "Have you noticed how the word 'intellectual' is used nowadays? There seems to be a new definition which certainly doesn't include Rutherford or Eddington or Dirac or Adrian[149] or me."'[150]

144 Mordell, 'Hardy's "A Mathematician's Apology"', p. 832.

145 Sydney Chapman (1888–1970): mathematician and geophysicist.

146 Mordell, 'Hardy's "A Mathematician's Apology"', p. 833.

147 Hersh & John-Steiner, *Loving + Hating Mathematics*, pp. 255–69; see also Hersh, 'Mathematical Menopause'.

148 Du Sautoy, 'Symmetry', p. 202.

149 Edgar Douglas Adrian, 1st Baron Adrian (1889–1977): physiologist; 1932 Nobel laureate in physiology or medicine.

150 Snow, *The Two Cultures*, § 1.

Second, the two cultures divide is between the arts and the sciences, but the *Apology* could hardly be blamed for a lack of expository work in the other sciences.

Third, on which side of the divide does mathematics actually lie? The *Apology*, and the views of many of the mathematicians in the later nineteenth and early twentieth centuries discussed above, taken together, arguably weaken the case for calling mathematics a science. Midgley suggested that the boundary between the 'two cultures' was hazy and that mathematics lay in the borderlands, suggesting that Hardy's view on the uselessness of mathematics 'seemed to demand for it a share in the peculiar kind of unworldly honour that was earmarked for the classics.'[151] As noted above (see p. 119), Ogilvy also wrote that mathematics resembles an art.

Fourth, whatever Hardy said in the opening lines, the *Apology* *is* an expositional work addressed to a general audience.

151 Midgley, 'The Use and Uselessness of Learning', p. 188.

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To avoid a surfeit of references, the term ‘mathematics’ itself is not indexed, nor are occasions when Hardy refers to himself in the first person in the *Apology*.

COLOPHON

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