

**GROUPS THAT TOGETHER WITH ANY TRANSFORMATION  
GENERATE REGULAR SEMIGROUPS OR IDEMPOTENT  
GENERATED SEMIGROUPS**

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ABSTRACT. Let  $a$  be a non-invertible transformation of a finite set and let  $G$  be a group of permutations on that same set. Then  $\langle G, a \rangle \setminus G$  is a subsemigroup, consisting of all non-invertible transformations, in the semigroup generated by  $G$  and  $a$ . Likewise, the conjugates  $a^g = g^{-1}ag$  of  $a$  by elements  $g \in G$  generate a semigroup denoted  $\langle a^g \mid g \in G \rangle$ . We classify the finite permutation groups  $G$  on a finite set  $X$  such that the semigroups  $\langle G, a \rangle$ ,  $\langle G, a \rangle \setminus G$ , and  $\langle a^g \mid g \in G \rangle$  are regular for all transformations of  $X$ . We also classify the permutation groups  $G$  on a finite set  $X$  such that the semigroups  $\langle G, a \rangle \setminus G$  and  $\langle a^g \mid g \in G \rangle$  are generated by their idempotents for all non-invertible transformations of  $X$ .

*Date:* 30 October 2009

*Key words and phrases:* Transformation semigroups, idempotent generated semigroups, regular semigroups, permutation groups, primitive groups, O’Nan-Scott Theorem.

*2010 Mathematics Subject Classification:* 20M20, 20M17, 20B30, 20B35, 20B15, 20B40

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## 1. INTRODUCTION

One of the fundamental aspects of the study of finite semigroups is the interplay between groups and idempotents. There are many examples among the most famous structural theorems in semigroup theory, such as, the Rees Theorem [24], or McAlister's  $P$ -Theorem [20, Part II, Theorem 2.6] (see also [22]), that show the large extent that the structure of a semigroup is shaped by a group acting in some way on an idempotent structure.

Another topic that has recently attracted a great deal of attention is a general problem that might be described as follows: *classify all groups  $G$  of permutations (possibly an adjective, such as, primitive, imprimitive, linear, rank  $k$ , added) of a set  $X$  such that  $G$  and an arbitrary singular mapping on  $X$  generate (or give rise, in some other sense to) a semigroup with a given property.* A mapping is *singular* if it is not a bijection, and hence, non-invertible. We offer the conjecture that the years to come will confirm this as a mainstream topic in semigroup theory. Such problems are considered, for instance, in [1], [3], [6], [23]. In this paper, we offer a contribution to this area by proving Theorems 1.1 and 1.2 below. Before stating these theorems we review some concepts and introduce some notation.

A *permutation group of degree  $n$*  is just a subgroup of  $\mathcal{S}_n$ . Recall that an element  $a$  of a semigroup  $S$  is said to be an *idempotent* if  $aa = a$ . We say that  $S$  is *idempotent generated* if it is generated by the set of its idempotent elements. An element  $a$  is *regular* if there is some  $b \in S$  such that  $a = aba$ . If  $a$  is an idempotent, then  $a = aaa$ , which shows that idempotent elements are always regular. A semigroup  $S$  is said to be *regular* if all its elements are regular. The symbols  $\mathcal{T}_n$ ,  $\mathcal{A}_n$ , and  $\mathcal{S}_n$  denote the semigroup of all transformations, the group of all invertible even transformations, and the group of all invertible transformations of the finite set  $\{1, \dots, n\}$ , respectively. The permutation groups that appear in the following theorems are described immediately before Theorem 2.7. If  $a$  is a transformation, and  $g$  is an invertible transformation, then  $a^g$  denotes the conjugate  $g^{-1}ag$  of  $a$  by  $g$ . If  $Y$  is a subset of  $\mathcal{T}_n$ , then  $\langle Y \rangle$  denotes the semigroup generated by  $Y$ .

The main theorems of our paper are the following.

**Theorem 1.1.** *If  $n \geq 1$  and  $G$  is a subgroup of  $\mathcal{S}_n$ , then the following are equivalent:*

- (i) *The semigroup  $\langle G, a \rangle \setminus G$  is idempotent generated for all  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$ .*
- (ii) *The semigroup  $\langle a^g \mid g \in G \rangle$  is idempotent generated for all  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$ .*
- (iii) *One of the following is valid for  $G$  and  $n$ :*
  - (a)  $n = 5$  and  $G \cong \text{AGL}(1, 5)$ ;
  - (b)  $n = 6$  and  $G \cong \text{PSL}(2, 5)$  or  $\text{PGL}(2, 5)$ ;
  - (c)  $G = \mathcal{A}_n$  or  $\mathcal{S}_n$ .

**Theorem 1.2.** *If  $n \geq 1$  and  $G$  is a subgroup of  $\mathcal{S}_n$ , then the following are equivalent:*

- (i) *The semigroup  $\langle G, a \rangle$  is regular for all  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$ .*
- (ii) *The semigroup  $\langle G, a \rangle \setminus G$  is regular for all  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$ .*
- (iii) *The semigroup  $\langle a^g \mid g \in G \rangle$  is regular for all  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$ .*
- (iv) *One of the following is valid for  $G$  and  $n$ :*
  - (a)  $n = 5$  and  $G \cong C_5$ ,  $D_5$ , or  $\text{AGL}(1, 5)$ ;
  - (b)  $n = 6$  and  $G \cong \text{PSL}(2, 5)$  or  $\text{PGL}(2, 5)$ ;
  - (c)  $n = 7$  and  $G \cong \text{AGL}(1, 7)$ ;

- (d)  $n = 8$  and  $G \cong \text{PGL}(2, 7)$ ;
- (e)  $n = 9$  and  $G \cong \text{PSL}(2, 8)$  or  $\text{PTL}(2, 8)$ ;
- (f)  $G = \mathcal{A}_n$  or  $\mathcal{S}_n$ .

The proofs of Theorems 1.1 and 1.2 will be given in Sections 3 and 4.

The line of research that this paper extends was initiated by Howie in [11]. Howie showed that the semigroup  $\langle \mathcal{S}_n, a \rangle \setminus \mathcal{S}_n$  is idempotent generated and regular whenever  $a$  is a singular transformation of  $\mathcal{T}_n$  with image of size  $n - 1$ . Later Symons [27] and Levi and McFadden [14] proved the following generalization. For a singular transformation  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$ , the semigroups  $\langle a^g \mid g \in \mathcal{S}_n \rangle$  and  $\langle \mathcal{S}_n, a \rangle \setminus \mathcal{S}_n$  coincide; in addition, this semigroup is idempotent generated and regular (see Lemma 2.1). Later Levi [15] showed that this last theorem remains true if we replace  $\mathcal{S}_n$  by the alternating group  $\mathcal{A}_n$ . Another related result is a theorem of McAlister stating, for an idempotent  $e$  with image of size  $n - 1$  and for a permutation group  $G \leq \mathcal{S}_n$ , that the semigroup  $\langle G, e \rangle$  is regular (see [21, Theorem 3.10]).

Theorems 1.1 and 1.2 generalize the results of Levi and McFadden [14, 15] referred to above, and also provide a converse.

The class of semigroups of the form  $\langle G, a \rangle \setminus G$  where  $G \leq \mathcal{S}_n$  and  $a \in \mathcal{T}_n$  is related to the class of  $\mathcal{S}_n$ -normal semigroups introduced in [26]. For a semigroup  $S \leq \mathcal{T}_n$  define

$$N(S) = \{g \in \mathcal{S}_n \mid S^g = S\},$$

where  $S^g = \{a^g \mid a \in S\}$ . It is clear that each element of  $N(S)$  induces an automorphism of  $S$ , and this gives rise to a homomorphism from  $N(S)$  into  $\text{Aut } S$ . If the kernel of this homomorphism is trivial, then we can consider  $N(S)$  as a subgroup of  $\text{Aut } S$ . A semigroup  $S \leq \mathcal{T}_n$  is said to be  $\mathcal{S}_n$ -normal if  $N(S) = \mathcal{S}_n$ . Schreier [25] and Mal'cev [18], more or less explicitly, proved, for a semigroup  $S$  containing all the constant mappings, that  $\text{Aut } S = N(S)$ . The class of  $\mathcal{S}_n$ -normal semigroups was characterized in [14] as follows: a semigroup  $S \leq \mathcal{T}_n \setminus \mathcal{S}_n$  is  $\mathcal{S}_n$ -normal if and only if

$$S = \bigcup_{\alpha \in S} \langle \mathcal{S}_n, \alpha \rangle \setminus \mathcal{S}_n.$$

This serves as a further incentive to study the class of semigroups  $\langle G, a \rangle \setminus G$  where  $G \leq \mathcal{S}_n$  and  $a \in \mathcal{T}_n$ .

The proofs of our main theorems rely on techniques from the theory of permutation groups and on explicit machine computations. In Section 2 we introduce a new property for permutation groups, namely the universal transversal property (see Definition 2.3). We refer to groups satisfying this property as universal transversal groups. We show in Lemma 3.1 that the assertions in Theorem 1.1(i)–(ii) and in Theorem 1.2(ii)–(iii) imply that  $G$  satisfies the universal transversal property. (It is not too hard to show that Theorem 1.2(ii) follows from Theorem 1.2(i); see the proof of Theorem 1.2 in Section 3.) The remaining part of Theorem 1.1 is verified by first giving an explicit list of the permutation groups that satisfy the universal transversal property (Theorem 2.7), and then verifying one-by-one which of them gives rise to an idempotent generated semigroup. The converse of Theorem 1.2 is proved using the results of [13]. The computations in the last stage of the proof were often carried out using the computational algebra system GAP [9]. Full details of these computations are available on the paper's companion webpage [2]; see also Section 5.

As mentioned in the previous paragraph, the proofs of the two main theorems are partly based on the classification of the universal transversal groups, given in Theorem 2.7. The action of a universal transversal group relates transversals and partitions in some way, and so the class of universal transversal groups is closely related to the class of synchronizing groups. In fact it is shown in Lemma 4.1 that a universal transversal group is synchronizing (see Section 4 for the definition). The class of synchronizing groups has recently been a very active research topic in the theory of finite permutation groups; see [1], [3], [6], and [23]. As synchronizing groups are primitive, it follows that universal transversal groups are too. It has been known for a long time that primitive subgroups of  $\mathcal{S}_n$  that do not contain  $\mathcal{A}_n$  must be small in comparison to the size of  $\mathcal{S}_n$ ; the estimate which is most useful for us is given by Maróti [19]. On the other hand, the nature of universal transversal groups imply that they must be large, and we use these two facts to classify them completely.

As Maróti's theorem uses the classification of finite simple groups, our main results also depend on the classification. Of course, the size of a primitive permutation group can be bounded independently of the classification; see the introduction of [19] for references. However, these bounds were not sufficiently practical for the purposes of proving Theorems 1.1 and 1.2.

The structure of the paper is as follows. In Section 2 we establish the connection between the classes of regular semigroups, idempotent generated semigroups, and universal transversal groups. The classification of universal transversal groups is stated without proof in the same section (Theorem 2.7). Assuming that we know the complete list of universal transversal groups, we prove the main theorems of the paper in Section 3. The proof of Theorem 2.7 is given in Section 4. Finally, in Section 5, we describe in more detail the machine calculations that were used in the proofs of our results.

We refer the reader to [12] for further information on the fundamentals of semigroup theory.

## 2. SEMIGROUPS AND THE UNIVERSAL TRANSVERSAL PROPERTY

The main objective of this paper is to investigate the semigroups  $\langle G, a \rangle \setminus G$  and  $\langle a^g \mid g \in G \rangle$  where  $G$  is a subgroup of  $\mathcal{S}_n$  and  $a$  is a non-invertible transformation in  $\mathcal{T}_n$ . Since, for  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$  and  $g \in \mathcal{S}_n$ , the element  $a^g$  is not invertible, we obtain that  $\langle a^g \mid g \in G \rangle \leq \langle G, a \rangle \setminus G$ . The next result states that certain semigroups arising from alternating and symmetric groups are regular and idempotent generated.

**Lemma 2.1.** *Let  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$  and let  $S = \langle a^g \mid g \in \mathcal{S}_n \rangle$ . Then the following hold:*

- (i)  $S = \langle \mathcal{S}_n, a \rangle \setminus \mathcal{S}_n = \langle \mathcal{A}_n, a \rangle \setminus \mathcal{A}_n = \langle a^g \mid g \in \mathcal{A}_n \rangle$ ;
- (ii)  $S$  is idempotent generated;
- (iii)  $S$  is regular.

*Proof.* It is easy to see, for  $b \in S$  and  $g \in \mathcal{S}_n$ , that  $b^g \in S$ , which shows that the semigroup  $S$  is  $\mathcal{S}_n$ -normal. Thus assertions (ii) and (iii) follow from [14, Theorem 6 and Proposition 9]. That  $S = \langle \mathcal{S}_n, a \rangle \setminus \mathcal{S}_n$  is noted after [14, Proposition 4], while  $S = \langle g^{-1}ag \mid g \in \mathcal{A}_n \rangle$  follows from [15, Proposition 6]. Therefore

$$S = \langle \mathcal{S}_n, a \rangle \setminus \mathcal{S}_n \geq \langle \mathcal{A}_n, a \rangle \setminus \mathcal{A}_n \geq \langle g^{-1}ag \mid g \in \mathcal{A}_n \rangle = S$$

which gives that the two inequalities in the previous displayed line are, in fact, equalities, and so (i) holds.  $\square$

The following result is noticed by McAlister in the proof of [21, Lemma 2.2].

**Lemma 2.2.** *For  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$  and  $G \leq \mathcal{S}_n$ , the semigroups  $\langle G, a \rangle \setminus G$  and  $\langle a^g \mid g \in G \rangle$  have the same set of idempotents. Consequently, if  $\langle G, a \rangle \setminus G$  is idempotent generated, then  $\langle G, a \rangle \setminus G = \langle a^g \mid g \in G \rangle$ .*

*Proof.* For a semigroup  $S$ , let  $E(S)$  denote the set of idempotents in  $S$ . Set  $S_1 = \langle a^g \mid g \in G \rangle$  and  $S_2 = \langle G, a \rangle \setminus G$ . As noted above,  $S_1 \leq S_2$ , and so we are only required to prove that  $E(S_2) \subseteq E(S_1)$ . Every element of the semigroup  $S_2$  can be written as  $g_1 a g_2 a \cdots g_n a g_{n+1}$  where  $g_i \in G$ . Let  $u = g_1 a g_2 a g_3 \cdots g_n a g_{n+1}$  be an idempotent of  $S_2$ . Then

$$u = a^{g_1^{-1}} a^{(g_1 g_2)^{-1}} a^{(g_1 g_2 g_3)^{-1}} \cdots a^{(g_1 g_2 g_3 \cdots g_n)^{-1}} (g_1 \cdots g_{n+1}).$$

Write  $u = vg$ , where  $g = g_1 \cdots g_{n+1}$  and  $v \in S_1$ . Now, as  $G$  is a finite group, there exists  $n \geq 1$  such that  $g^n$  is the identity and, as  $u = vg$  is idempotent, we have

$$\begin{aligned} vg &= (vg)^n \\ &= v(gvg^{-1})(g^2vg^{-2}) \cdots (g^{n-1}vg^{-n+1})g^n \\ &= v(gvg^{-1})(g^2vg^{-2}) \cdots (g^{n-1}vg^{-n+1}) \in S_1. \end{aligned}$$

Hence  $u = vg$  is an idempotent of  $S_1$ , and so  $E(S_2) \subseteq E(S_1)$ , as claimed.

To prove the second assertion, suppose that  $E$  denotes the set of idempotents in  $S_2$  and assume that  $\langle E \rangle = S_2$ . The first assertion of the lemma implies that  $E \subseteq S_1$ , and so  $S_2 = \langle E \rangle \leq S_1$ . As  $S_1 \leq S_2$ , the equality  $S_1 = S_2$  follows.  $\square$

Next we define the *universal transversal property* for permutation groups. Let  $P$  be a partition of a set  $X$ . Recall that a subset  $I$  of  $X$  is called a *transversal* for  $P$  if every class of  $P$  contains precisely one element of  $I$ . If  $X$  is a set and  $a$  is a transformation of  $X$ , then the image of an element  $\alpha \in X$  under  $a$  is denoted by  $\alpha a$ . If  $Y$  is a subset of  $X$ , then we may consider the image  $Ya = \{\alpha a \mid \alpha \in Y\}$  of  $Y$  under  $a$ .

**Definition 2.3.** A permutation group  $G$  of degree  $n$  is said to have the *universal transversal property* if for every subset  $I$  of  $\{1, 2, \dots, n\}$  and every partition  $P$  of  $\{1, 2, \dots, n\}$  with  $|I|$  classes, there exists  $g \in G$  such that  $Ig$  is a transversal for  $P$ .

In the following examples we describe a group that satisfies the universal transversal property, and another that does not. Recall that if  $G$  is a permutation group acting on a set  $X$ , then, for  $\alpha \in X$ , the set  $\{\alpha g \mid g \in G\}$  is said to be a *G-orbit* and is denoted by  $\alpha G$ . The set of  $G$ -orbits form a partition of  $X$  and  $G$  is called *transitive* if  $X$  is a  $G$ -orbit. The permutation group  $G$  acts on the set of subsets of  $X$  (defined above) and we may consider the  $G$ -orbit of a subset  $Y \subseteq X$ .

**Example 2.4.** Set  $G = \langle (12345) \rangle$ . Then  $G$  is a cyclic group with order 5, and we claim that  $G$  satisfies the universal transversal property. For  $r \in \{1, 4, 5\}$ , the group  $G$  is transitive on the set of subsets containing precisely  $r$ -elements. Hence we only need to prove this claim for transversals and partitions containing either two or three members. The  $G$ -orbits on the set of subsets with two elements are

$$\begin{aligned} \{1, 2\}G &= \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\} \quad \text{and} \\ \{1, 3\}G &= \{\{1, 3\}, \{2, 4\}, \{3, 5\}, \{4, 1\}, \{5, 2\}\}. \end{aligned}$$

Suppose that  $P$  is a partition of  $\{1, \dots, 5\}$  with two classes such that none of the elements in one of the orbits is a transversal for  $P$ . Then every pair of elements

in the orbit must lie in the same class of  $P$ , which implies that  $P$  contains only one class, a contradiction. The  $G$ -orbits on the set of subsets with three elements are  $\{1, 2, 3\}G$  and  $\{1, 2, 4\}G$ . An argument similar to the one above, shows that a partition that has no transversal in one of these orbits can have at most two classes. Hence  $G$  must have the universal transversal property.

**Example 2.5.** Set  $G = \langle (1\ 2\ 3\ 4\ 5\ 6\ 7) \rangle$ . Then  $G$  is a cyclic group with order 7. In contrast with the previous example,  $G$  does not satisfy the universal transversal property. Indeed, an easy calculation shows that the  $G$ -orbit

$$\{\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{4, 5, 6\}, \{5, 6, 7\}, \{1, 6, 7\}, \{1, 2, 7\}\}$$

contains no transversal for the partition  $\{\{1\}, \{2, 3, 4, 6, 7\}, \{5\}\}$ .

Recall that the *kernel* of a transformation  $a \in \mathcal{T}_n$  is the equivalence relation  $\{(\alpha, \beta) \mid \alpha a = \beta a\}$  and is denoted by  $\ker a$ . The image of  $a \in \mathcal{T}_n$  is denoted by  $\text{im } a$ . The rank of a transformation  $a$  is defined as  $|\text{im } a|$ . Following [13], for a given  $G \leq \mathcal{S}_n$  and  $a \in \mathcal{T}_n$ , we set

$$K_G(a) = \{g \in G \mid \text{rank } a = \text{rank } aga\}.$$

It is straightforward to verify (and also noted before [13, Theorem 2.3]) that  $g \in K_G(a)$  if and only if  $g$  maps  $\text{im } a$  into a transversal of  $\ker a$ .

**Lemma 2.6.** *Let  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$  and let  $G \leq \mathcal{S}_n$ . Then  $G$  satisfies the universal transversal property if and only if  $K_G(a) \neq \emptyset$  for all  $a \in \mathcal{T}_n$ .*

*Proof.* Note that  $G$  satisfies the universal transversal property if and only if for all  $a \in \mathcal{T}_n$  there is some  $g \in G$  such that  $\text{im } ag$  is a transversal for  $\ker a$ ; that is  $g \in K_G(a)$ . Hence  $G$  satisfies the universal transversal property if and only if  $K_G(a) \neq \emptyset$  for all  $a \in \mathcal{T}_n$ .  $\square$

The main results of this paper rely on the classification of universal transversal groups, given in the following theorem. The permutation groups that appear in this theorem are considered in their natural actions. The group  $\text{AGL}(1, p)$  acts on the  $p$  vectors of a one-dimensional vector space over the field of  $p$  elements, and the groups  $C_5$ ,  $D_5$  are considered as subgroups of  $\text{AGL}(1, 5)$ . The projective groups  $\text{PGL}$ ,  $\text{PSL}$ , and  $\text{PTL}$  are viewed as permutation groups acting on the set of projective points (that is, the set of one-dimensional subspaces) of their natural module.

**Theorem 2.7.** *A subgroup  $G$  of  $\mathcal{S}_n$  has the universal transversal property if and only if one of the following is valid:*

- (i)  $n = 5$  and  $G \cong C_5$ ,  $D_5$ , or  $\text{AGL}(1, 5)$ ;
- (ii)  $n = 6$  and  $G \cong \text{PSL}(2, 5)$  or  $\text{PGL}(2, 5)$ ;
- (iii)  $n = 7$  and  $G \cong \text{AGL}(1, 7)$ ;
- (iv)  $n = 8$  and  $G \cong \text{PGL}(2, 7)$ ;
- (v)  $n = 9$  and  $G \cong \text{PSL}(2, 8)$  or  $\text{PTL}(2, 8)$ ;
- (vi)  $G = \mathcal{A}_n$  or  $\mathcal{S}_n$ .

The proof of Theorem 2.7 will be given in Section 4.

3. THE PROOFS OF THE MAIN RESULTS

In this section we prove Theorems 1.1 and 1.2. We start with a lemma that links the universal transversal property with the classes of idempotent generated semigroups and regular semigroups. This lemma essentially allows us to prove one direction of the main theorems.

**Lemma 3.1.** *Let  $G$  be a subgroup of  $\mathcal{S}_n$  such that one of the following properties holds:*

- (i)  $\langle G, a \rangle \setminus G$  is idempotent generated for all  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$ ;
- (ii)  $\langle G, a \rangle \setminus G$  is regular for all  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$ ;
- (iii)  $\langle a^g \mid g \in G \rangle$  is idempotent generated for all  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$ .
- (iv)  $\langle a^g \mid g \in G \rangle$  is regular for all  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$ .

*Then  $G$  has the universal transversal property.*

*Proof.* Let us assume by contradiction that  $G$  does not have the universal transversal property, and show that assertions (i)–(iv) are not valid. By Lemma 2.6, there is some  $a \in \mathcal{T}_n$  such that  $K_G(a) = \emptyset$ . We note that  $\langle a^g \mid g \in G \rangle \leq \langle G, a \rangle \setminus G$  and that  $a \in \langle a^g \mid g \in G \rangle \cap \langle G, a \rangle \setminus G$ .

Let  $b$  be an idempotent in  $\langle G, a \rangle \setminus G$  and hence in  $\langle a^g \mid g \in G \rangle$  by Lemma 2.2. Then  $b$  is of the form  $g_1 a g_2 a \cdots g_{k-1} a g_k$  where  $g_1, \dots, g_k \in G$ . We claim that  $k \geq 3$ . Indeed if  $g_1 a g_2$  is an idempotent with some  $g_1, g_2 \in G$ , then  $g_1 a g_2 g_1 a g_2 = g_1 a g_2$  and so  $a g_1 g_2 a = a$ . In particular,  $\text{rank } a g_1 g_2 a = \text{rank } a$ , which is impossible as  $K_G(a) = \emptyset$ . Hence  $b = g_1 a g_2 a \cdots g_{k-1} a g_k$  with  $k \geq 3$ , as claimed. Note that  $\text{rank } b \leq \text{rank } a g_2 a$  and, as  $g_2 \notin K_G(a)$ , that  $\text{rank } a g_2 a < \text{rank } a$ . Therefore  $\text{rank } b < \text{rank } a$ , and the element  $a$  is not a member of the semigroup generated by the idempotents of  $\langle G, a \rangle \setminus G$  or  $\langle a^g \mid g \in G \rangle$ . Thus assertions (i) and (iii) do not hold.

To prove that assertions (ii) and (iv) fail, we show that  $a$  is not a regular element of  $\langle G, a \rangle \setminus G$ . Suppose as above that  $a$  is regular and there is  $b \in \langle G, a \rangle \setminus G$  such that  $a = aba$ . Then  $\text{rank } ab = \text{rank } a$ . In addition,  $ab = abab$ , and so  $ab$  is an idempotent. We obtain a contradiction, as, by the previous paragraph, the semigroup  $\langle G, a \rangle \setminus G$  has no idempotents with the same rank as  $a$ . That is,  $a$  is not a regular element of  $\langle G, a \rangle \setminus G$ , and, since  $\langle a^g \mid g \in G \rangle \leq \langle G, a \rangle \setminus G$ , it is not a regular element of  $\langle a^g \mid g \in G \rangle$ . Therefore  $\langle G, a \rangle \setminus G$  is not a regular semigroup, and neither is its subsemigroup  $\langle a^g \mid g \in G \rangle$ .  $\square$

We can now prove Theorem 1.1. The symbol  $[\alpha_1, \alpha_2, \dots, \alpha_n]$  denotes the element of  $\mathcal{T}_n$  that maps  $1 \mapsto \alpha_1, 2 \mapsto \alpha_2, \dots, n \mapsto \alpha_n$ .

*Proof of Theorem 1.1.* If  $\langle G, a \rangle \setminus G$  is idempotent generated, then, by Lemma 2.2,  $\langle a^g \mid g \in G \rangle = \langle G, a \rangle \setminus G$ , and so  $\langle a^g \mid g \in G \rangle$  is also idempotent generated. Thus statement (i) implies statement (ii).

Let us next show that statement (ii) implies statement (iii). By assumption,  $\langle a^g \mid g \in G \rangle$  is idempotent generated for all  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$ . Hence Lemma 3.1 implies that  $G$  satisfies the universal transversal property. By Theorem 2.7, it suffices to show, for  $G \in \{C_5, D_5, \text{AGL}(1, 7), \text{PGL}(2, 7), \text{PSL}(2, 8), \text{PTL}(2, 8)\}$ , that there exists some  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$  such that the semigroup  $\langle a^g \mid g \in G \rangle$  is not generated by idempotents. Using the GAP computational algebra system, it is possible to show that the required transformations  $a$  are  $[1, 3, 2, 2, 2]$ ,  $[1, 2, 3, 3, 3]$ ,  $[1, 2, 3, 3, 3, 3, 3]$ ,  $[6, 2, 3, 4, 6, 6, 6, 6]$ ,  $[1, 2, 3, 5, 4, 5, 4, 4, 5]$ , and  $[1, 2, 3, 5, 4, 5, 4, 4, 5]$ , respectively; see Section 5 and [2] for further description of the computations.

The assertion that statement (iii) implies statement (i) is verified as follows. The groups  $\mathcal{S}_n$  and  $\mathcal{A}_n$  satisfy statement (i), by Lemma 2.1, so we may assume that  $G$  is one of the groups  $\text{AGL}(1, 5)$ ,  $\text{PSL}(2, 5)$ ,  $\text{PGL}(2, 5)$ . We used the computational algebra package `GAP` to verify, for these groups, that for all  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$  the semigroup  $\langle G, a \rangle \setminus G$  is idempotent generated. See Section 5 and [2] for the details.  $\square$

Let us now prove Theorem 1.2.

*Proof of Theorem 1.2.* First we prove that (i) implies (ii). It suffices to prove that if  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$  such that  $\langle G, a \rangle$  is regular, then  $\langle G, a \rangle \setminus G$  is regular. Assume that  $\langle G, a \rangle$  is regular for some  $G \leq \mathcal{S}_n$  and for some  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$ . Then for each  $u \in \langle G, a \rangle \setminus G$  there is some  $v \in \langle G, a \rangle$  with  $u = uvu$ . If  $v \notin G$ , then  $u$  is a regular element of  $\langle G, a \rangle \setminus G$ , and so we may assume that  $v \in G$ . Then  $vvv \notin G$  and  $u(vvv)u = (uvu)vu = uvu = u$ . Thus  $u$  is regular in  $\langle G, a \rangle \setminus G$  also in this case, and so  $\langle G, a \rangle \setminus G$  is a regular semigroup, as claimed.

Next we prove that assertion (ii) implies (iii). Suppose that  $G$  is a permutation group such that  $\langle G, a \rangle \setminus G$  is regular for all  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$ . Then, by Lemma 3.1,  $G$  satisfies the universal transversal property, and so  $K_G(a) \neq \emptyset$  for all  $a \in \mathcal{T}_n$ . Let  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$ , and let  $b \in \langle a^g \mid g \in G \rangle$ . Then  $K_G(b) \neq \emptyset$ , and hence [13, Theorem 2.3] implies that  $b$  is regular in  $\langle b^g \mid g \in G \rangle$ . As  $\langle b^g \mid g \in G \rangle \leq \langle a^g \mid g \in G \rangle$ , we find that  $b$  is regular in  $\langle a^g \mid g \in G \rangle$ , and so  $\langle a^g \mid g \in G \rangle$  is a regular semigroup.

The fact that (iii) implies (iv) is a consequence of Lemma 3.1 and Theorem 2.7.

Finally, we prove that (iv) implies (i). Let  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$ . We are required to show, for  $b \in \langle G, a \rangle$ , that  $b$  is regular in  $\langle G, a \rangle$ . As  $G$  has the universal transversal property, Lemma 2.6 gives that  $K_G(b) \neq \emptyset$ . Thus, by [13, Theorem 2.3], the element  $b$  is regular in  $\langle G, b \rangle$ . As  $\langle G, b \rangle \leq \langle G, a \rangle$ , we find that  $b$  is regular in  $\langle G, a \rangle$ , as claimed.  $\square$

#### 4. THE CLASSIFICATION OF UNIVERSAL TRANSVERSAL GROUPS

The proof of Theorem 2.7 is given in this section. It can be verified using the computational algebra system `GAP` [9] that the groups  $C_5$ ,  $D_5$ ,  $\text{AGL}(1, 5)$ ,  $\text{PSL}(2, 5)$ ,  $\text{PGL}(2, 5)$ ,  $\text{AGL}(1, 7)$ ,  $\text{PGL}(2, 7)$ ,  $\text{PSL}(2, 8)$ ,  $\text{PGL}(2, 8)$  listed in the theorem satisfy the universal transversal property; full details of the computation are available on the companion webpage [2] (see also Section 5). A permutation group acting on a set  $X$  is said to be *k-homogeneous* if it is transitive on the set of subsets of  $X$  with size  $k$ . The alternating group  $\mathcal{A}_n$  is *k-homogeneous* for  $k = 1, 2, \dots, n$ ; see [8, Exercise 2.1.4]. Thus the *k-homogeneous* groups  $\mathcal{A}_n$  and  $\mathcal{S}_n$  have the universal transversal property, and so the proof of one direction of Theorem 2.7 is concluded.

Synchronizing groups were first introduced in [3] and a combinatorial characterization was given in [1]. A permutation group  $G \leq \mathcal{S}_n$  is said to be *synchronizing* if for every non-trivial partition  $P$  (a partition is said to be *non-trivial* if it has at least 2 and at most  $n - 1$  blocks) of  $\{1, \dots, n\}$  and every transversal  $S$  of  $P$ , there exists  $g \in G$  such that  $Sg$  is not a transversal for  $P$ . Recall that a permutation group is said to be *primitive* if no non-trivial partition of the underlying set is invariant under the group action; see also [8, Section 1.5]. If  $G$  preserves a partition  $P$  of  $\{1, \dots, n\}$ , then any image of a transversal of  $P$  is again a transversal. Hence a synchronizing group is transitive and primitive (see also [23, Introduction]).

**Lemma 4.1.** *Every universal transversal group is synchronizing, and hence such a group is transitive and primitive.*



*Proof.* Suppose that  $G$  is a universal transversal group acting on a set  $X$ . Let  $P$  be a non-trivial partition of  $X$  and let  $S$  be a transversal for  $P$ . Let  $\alpha, \beta \in X$  be two distinct elements in the same block of  $P$ . Let  $P_1$  be a partition of  $X$  such that  $\{\alpha\}, \{\beta\} \in P_1$  and  $|P| = |P_1|$ . Now, by assumption, there exists  $g \in G$  such that  $Sg$  is a transversal for  $P_1$  and hence  $\alpha, \beta \in Sg$ . Thus  $Sg$  is not a transversal for  $P$ , which shows that  $G$  is synchronizing. Therefore  $G$  is transitive and primitive.  $\square$

We note that it is possible to prove that a universal transversal group is transitive and primitive without using the concept of synchronizing groups. However, we decided to include the proof above, as synchronizing groups will play some further role in this paper.

A subgroup of  $\mathcal{S}_n$  is said to be *proper* if it does not contain  $\mathcal{A}_n$ . Next we prove that a proper primitive group of large enough degree does not satisfy the universal transversal property. Before proving this result we introduce some terminology. Let  $r \in \mathbb{N}$ . We say that a partition  $P$  of a set is *r-singular* if it has  $r + 1$  classes of which contain exactly 1 element. The number of *r-singular* partitions of  $\{1, \dots, n\}$  is clearly  $\binom{n}{r}$ . Further, if  $R$  is a subset of  $\{1, \dots, n\}$  with  $r + 1$  elements, then the number of *r-singular* partitions  $P$  of  $\{1, \dots, n\}$  such that  $R$  is a transversal for  $P$  is  $r + 1$ .

**Lemma 4.2.** *If  $G$  is a proper primitive group with degree at least 47, then  $G$  does not satisfy the universal transversal property.*

*Proof.* It is known that proper primitive groups have ‘small’ orders. More precisely, in [19, Corollary 1.1] it is proved that if  $G \leq \mathcal{S}_n$  is a proper primitive group, then

$$(1) \quad |G| < 50n^{\sqrt{n}}.$$

Thus the strategy of the proof is to argue that universal transversal groups must have order at least  $50n^{\sqrt{n}}$ . The first step is to find a lower bound for the order of a group with the universal transversal property.

Suppose  $R$  is an  $(r + 1)$ -element set. Then the orbit of  $R$  under  $G$  must contain a transversal for every *r-singular* partition of  $\{1, \dots, n\}$ . As noted above, the number of *r-singular* partitions of  $\{1, \dots, n\}$  is  $\binom{n}{r}$  and each  $(r + 1)$ -element set is a transversal for exactly  $(r + 1)$  *r-singular* partitions of  $\{1, \dots, n\}$ . Hence if  $G$  has the universal transversal property, then

$$(2) \quad |G|(r + 1) \geq \binom{n}{r}$$

must hold for all  $r \in \{1, \dots, n - 1\}$ , and hence

$$(3) \quad \binom{n}{r} \leq 50n^{\sqrt{n}}(r + 1).$$

We complete the proof by showing that (3) fails to hold for  $n \geq 47$  with  $r = n/2$  when  $n$  is even and  $r = (n + 1)/2$  when  $n$  is odd. That is, we show that

$$(4) \quad 50(2r)^{\sqrt{2r}}(r + 1) < \binom{2r}{r}$$

and that

$$(5) \quad 50(2r + 1)^{\sqrt{2r+1}}(r + 1) < \binom{2r + 1}{r}$$

holds for sufficiently large  $r$ . We only prove (4), as the proof of (5) is very similar. Let  $A(r)$  and  $B(r)$  denote the left-hand side and the right-hand side of (4), respectively. We use induction on  $r$ . It is easy to compute that  $A(r) < B(r)$  for  $r \in \{24, \dots, 46\}$ . Assume that  $r \geq 46$ , that  $A(r) < B(r)$ , and let us show that  $A(r+1) < B(r+1)$ . We claim that  $A(r+1)/A(r) < B(r+1)/B(r)$  which implies immediately that  $A(r+1) < B(r+1)$ . First, note that  $B(r+1)/B(r) = (2r+1)(2r+2)/(r+1)^2 = 2(2r+1)/(r+1)$ , and that

$$\frac{A(r+1)}{A(r)} = \frac{50(2r+2)^{\sqrt{2r+2}}(r+2)}{50(2r)^{\sqrt{2r}}(r+1)} \leq \frac{2(2r+2)^{\sqrt{2r+2}}}{(2r)^{\sqrt{2r}}}.$$

Hence it suffices to show that

$$(6) \quad \frac{(2r+2)^{\sqrt{2r+2}}}{(2r)^{\sqrt{2r}}} < \frac{2r+1}{r+1}.$$

It is easy to see that (6) holds for  $r = 46$ . The fact that it holds for  $r \geq 47$  follows from the observation that the left-hand side of (6) is decreasing, while the right-hand side is increasing. The latter of these claims is trivial. For the former, the derivative of the function  $x \mapsto (2x+2)^{\sqrt{2x+2}}/2x^{\sqrt{2x}}$  is

$$x \mapsto C(x) \left( \frac{\sqrt{x}\sqrt{x+1} \log(x+1) + \sqrt{x}\sqrt{x+1}(\log 2 + 2)}{x^{(2\sqrt{2}\sqrt{x+1})/2}(\sqrt{2x} + \sqrt{2})2^{\sqrt{2}\sqrt{x}}} - \frac{(x+1) \log x + (\log 2 + 2)(x+1)}{x^{(2\sqrt{2}\sqrt{x+1})/2}(\sqrt{2x} + \sqrt{2})2^{\sqrt{2}\sqrt{x}}} \right),$$

where  $C(x) = (x+1)^{\sqrt{2}\sqrt{x+1}}2^{\sqrt{2}\sqrt{x+1}}$ . Since  $\sqrt{x}\sqrt{x+1} \log(x+1) < (x+1) \log x$  and  $\sqrt{x}\sqrt{x+1} < (x+1)$ , we obtain that this derivative is negative, which shows that the left-hand side of (6) is decreasing, as claimed.  $\square$

As the previous lemma gives a practical upper bound for the degree of a proper universal transversal group, we could finish the classification of such groups using computer calculation only. However, using the structure theorem of finite primitive permutation groups and some elementary combinatorics, we can significantly reduce the amount of computer calculation that is required to prove Theorem 2.7. Primitive permutation groups are described by the O’Nan-Scott Theorem that divides these group into several classes. Statements of this theorem can be found in [8, Section 4.8] and in [5, Sections 4.4–4.5], while in [4] there is a detailed comparison of the different versions of the theorem that can be found in the literature. Since the order of a non-abelian finite simple group is at least 60, combining the O’Nan-Scott Theorem with the bound in Lemma 4.2 gives that a proper universal transversal group is either an almost simple group, an affine group, or a subgroup of a wreath product in product action.

A finite group is said to be *almost simple* if it has a unique minimal normal subgroup which is a non-abelian simple group. Almost simple primitive groups form a class of primitive groups in the O’Nan-Scott Theorem. If  $G$  is a permutation group acting on  $X$ , then the wreath product  $W = G \wr \mathcal{S}_n$  can be considered as a permutation group acting on the cartesian product  $X^n$ . A primitive group of product action type is a suitable subgroup of such a wreath product  $W$  in the case when  $G$  is an almost simple primitive group.

A primitive permutation group  $G$  is said to be *affine* if it has an abelian normal subgroup. Affine primitive groups can be characterized as follows. Let  $V$  be an  $n$ -dimensional vector space over a field  $\mathbb{F}_p$  of  $p$  elements for some prime  $p$ , and let  $H$  be a subgroup of the group  $\text{GL}(V)$  of invertible linear transformations of  $V$ . Every element  $v \in V$  defines a permutation  $\tau_v$  of  $V$  where  $\tau_v : u \mapsto u+v$  for all  $u \in V$ . The collection  $T$  of the  $\tau_v$  is a subgroup of the full symmetric group  $\text{Sym } V$  isomorphic to the additive group of  $V$ . Similarly, the elements of  $H$  can be considered as permutations of  $V$ , and so  $H$  can also be viewed as a subgroup of  $\text{Sym } V$ . It is easy to see that  $H$  normalizes  $T$ , and so their product  $TH$  is a subgroup of  $\text{Sym } V$ . As  $T$  is transitive, so is  $TH$ . Further,  $TH$  is primitive if and only if  $H$  is irreducible; that is, no non-trivial, proper subspace of  $V$  is invariant under  $H$ . In this case, as  $T$  is an abelian normal subgroup of  $TH$ , the primitive group  $TH$  is affine. Conversely, by [8, Theorem 4.7A], every affine primitive group is permutationally isomorphic to a group of the form  $TH$ . Permutational isomorphism is defined in [8, page 17]. Permutationally isomorphic groups are essentially the same except for the labeling of the points on which they act. In particular, the degree of an affine primitive group is a prime-power. For each prime-power  $p^k$  there is a largest affine group with degree  $p^k$  constructed as follows. Let  $V$  be the  $k$ -dimensional vector space over  $\mathbb{F}_p$  and let  $T$  denote the subgroup formed by the  $\tau_v$  as defined above. Then the group  $T\text{GL}(V)$  is called the *affine general linear group* and is denoted by  $\text{AGL}(k, p)$ . Every affine primitive group with degree  $p^k$  is a subgroup of  $\text{AGL}(k, p)$ .

**Lemma 4.3.** *If  $G$  is a permutation group of degree  $n$  with the universal transversal property, then one of the following must hold:*

- (i)  $n \leq 4$ ;
- (ii)  $G \leq \text{AGL}(1, p)$  with  $p \in \{5, 7\}$ ;
- (iii)  $G$  is an almost simple primitive group.

*Proof.* By Lemmas 4.1 and 4.2, a group with the universal transversal property is synchronizing, and hence it is primitive of degree at most 46. As noted above, a primitive group of such small degree is either almost simple, or a subgroup of a wreath product, or affine. As subgroups of wreath products in product action are non-synchronizing (see [23, Example 3.4]), they cannot have the universal transversal property, by Lemma 4.1. Since an affine primitive group of degree  $p^k$  is a subgroup of  $\text{AGL}(k, p)$ , and there is no affine group of degree 6, the statement of the theorem is valid whenever  $n \leq 7$ . Thus we may assume without loss of generality that  $n > 7$  and we are required to show that an affine primitive group with degree  $n$  does not have the universal transversal property.

Assume that  $G$  is a primitive group of affine type acting on a vector space  $V = \mathbb{F}_p^k$  for some prime  $p$  and integer  $k$ . A translate of a one-dimensional subspace in  $V$  is said to be a *line* and a translate of a 2-dimensional subspace is said to be a *plane*. The action of  $G$  on  $V$  preserves the set of lines and the set of planes. Suppose first that  $p$  is at least 5 and  $\dim V \geq 2$ . In this case a line has at least 5 elements and there are three vectors which do not lie on a common line. Choose 3 vectors  $v_1, v_2, v_3$  which do not lie on a common line and let  $P$  be the partition

$$\{\{v_1\}, \{v_2\}, \{v_3\}, V \setminus \{v_1, v_2, v_3\}\}.$$

Now let  $u_1, u_2, u_3, u_4$  be 4 vectors on a common line and set

$$S = \{u_1, u_2, u_3, u_4\}.$$

If  $g$  is an element of  $G$ , then the image  $Sg$  contains 4 vectors which lie on a common line. On the other hand, if  $R$  is a transversal for  $P$ , then  $R$  must contain  $v_1, v_2, v_3$  and these do not lie on a common line. Thus no image  $Sg$  of  $S$  can be a transversal of  $P$ , and so  $G$  does not have the universal transversal property.

Suppose now that either  $p = 2$  and  $\dim V \geq 4$  or  $p = 3$  and  $\dim V \geq 3$ . In this case we choose four vectors  $v_1, v_2, v_3$ , and  $v_4$  on a common plane and choose 5 vectors  $u_1, u_2, u_3, u_4, u_5$  so that no four of them lie on a common plane. If  $p = 3$ , then we may choose  $u_1 = 0, u_2 = b_1, u_3 = b_2, u_4 = b_3, u_5 = b_1 + b_2 + b_3$ , while if  $p = 2$  then  $u_1 = 0, u_2 = b_1, u_3 = b_2, u_4 = b_3$  and  $u_5 = b_4$  are suitable where  $b_1, b_2, \dots, b_k$  is a basis of  $V$ . Then an argument similar to the one above shows that no image of  $\{u_1, u_2, u_3, u_4, u_5\}$  by any element of  $G$  is a transversal for the partition

$$\{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, V \setminus \{v_1, v_2, v_3, v_4\}\}.$$

Assume now that  $\dim V = 1$  and let  $p \geq 11$ . We identify  $V$  with  $\mathbb{F}_p$ . Set

$$P = \{\{0\}, \{1\}, \{2\}, \mathbb{F}_p \setminus \{0, 1, 2\}\}$$

and

$$S = \{0, 1, 3, 4\}.$$

If  $g \in G$ , then there are  $a, b \in \mathbb{F}_p$  such that  $xg = (x + a)b = xb + ab$  for all  $x \in \mathbb{F}_p$ . Thus  $0g = ab, 1g = b + ab, 3g = 3b + ab, 4g = 4b + ab$ . If  $Sg = \{ab, b + ab, 3b + ab, 4b + ab\}$  is a transversal for  $P$ , then there are three elements  $x, y$ , and  $z$  of  $Sg$  such that  $x - y = y - z$ . However, using that  $p \geq 11$ , inspecting all 24 possibilities for  $x, y$ , and  $z$ , we find that it is not possible to choose such elements.

The remaining cases  $p = 2, \dim V = 3$  and  $p = 3, \dim V = 2$  can be handled as follows. Let  $G \leq \text{AGL}(3, 2)$ , then  $G$  can be viewed as a group acting on the 3-dimensional vector space  $V$  over  $\mathbb{F}_2$ . It is not hard to verify that the orbit of the subset

$$\{(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1)\}$$

does not contain a transversal for the partition

$$\{\{(0, 0, 0), (0, 0, 1)\}, \{(0, 1, 0), (0, 1, 1)\}, \{(1, 0, 0), (1, 0, 1)\}, \{(1, 1, 0)\}, \{(1, 1, 1)\}\}.$$

The last case is  $G \leq \text{AGL}(2, 3)$  and hand computation can show that the orbit of the subset

$$\{(0, 0), (0, 1), (0, 2)\}$$

does not contain a transversal for the partition

$$\{\{(0, 0)\}, \{(0, 1), (0, 2), (1, 2), (2, 1)\}, \{(1, 0), (2, 0), (1, 1), (2, 2)\}\}.$$

The proof is now complete.  $\square$

Now we are ready to prove Theorem 2.7.

*The proof of Theorem 2.7.* Suppose that  $G \leq \mathcal{S}_n$  is a proper primitive permutation group that satisfies the universal transversal property. We may assume that  $G$  is not listed in Theorem 2.7. A primitive group of degree 3 or 4 is either an alternating or a symmetric group. A primitive group of degree 5, is either an alternating or symmetric group, or listed in Theorem 2.7(i). Thus  $G$  must either be a subgroup of  $\text{AGL}(1, 7)$  or an almost simple group. Further, we proved in Lemma 4.2, that the order of  $G$  must satisfy (2) with  $r = \lfloor n/2 \rfloor$ . Using the primitive groups library of GAP, we found that there are 32 such groups and they are listed in the second

Deg	Group(s)	GAP id.	Set	Partition
7	PSL(3, 2)	5	{1, 2, 4}	{1}, {2, 3, 4, 7}, {5, 6}
	7 : 3	3	{1, 2, 4, 7}	{1}, {2}, {3}, {4, 5, 6, 7}
	$D_7$	2	{1, 3, 7}	{1}, {2}, {3, 4, 5, 6, 7}
8	PSL(2, 7)	4	{1, 2, 3, 5}	{1}, {2}, {3, 4, 5, 7}, {6, 8}
10	$\mathcal{A}_5, \mathcal{S}_5$	1, 2	{1, 2, 3, 5, 6}	{1}, \dots, {5}, {6, \dots, 10}
	PSL(2, 9), PGL(2, 9)	3, 4		
	$\mathcal{S}_6, M(10)$	5, 6		
	PTL(2, 9)	7		
11	PSL(2, 11)	5	{1, 2, 3, 5}	{1}, {2}, {3, \dots, 11}, {9}
	M(11)	6	{1, 2, 3, 4, 6}	{1}, {2}, {3}, {4, 5, 6, 7, 10, 11}, {8, 9}
12	M(12)	2	{1, \dots, 6}	{1, \dots, 6}, {7, 8}, {9}, \dots, {12}
	M(11)	1	{1, 2, 3, 4, 11, 12}	{1}, \dots, {5}, {6, \dots, 12}
	PGL(2, 11), PSL(2, 11)	4, 3	{1, 2, 3, 4, 6, 7}	{1}, \dots, {5}, {6, \dots, 12}
13	PSL(3, 3)	7	{1, \dots, 5, 7, 8}	{1}, \dots, {6}, {7, \dots, 13}
14	PGL(2, 13), PSL(2, 13)	2, 1	{1, \dots, 6, 9, 12}	{1}, \dots, {7}, {8, \dots, 14}
15	PSL(4, 2), $\mathcal{A}_7$	4, 1	{1, \dots, 6, 8, 12}	{1}, \dots, {7}, {8, \dots, 15}
17	PSL(2, $2^4$ ), PSL(2, $2^4$ ) : 4	6, 8	{1, \dots, 7, 11, 14}	{1}, \dots, {8}, {9, \dots, 17}
	PSL(2, $2^4$ ) : 2	7	{8, \dots, 14, 16, 17}	{1}, \dots, {8}, {9, \dots, 17}
18	PGL(2, 17)	2	{1, \dots, 8, 10, 11}	{1}, \dots, {9}, {10, \dots, 18}
21	P $\Sigma$ L(3, 4), PGL(3, 4), P $\Gamma$ L(3, 4)	5, 6 7	{1, \dots, 9, 11, 13}	{1}, \dots, {10}, {11, \dots, 21}
22	M(22), M(22) : 2	1, 2	{1, \dots, 10, 12, 15}	{1}, \dots, {11}, {12, \dots, 22}
23	M(23)	5	{1, \dots, 5, 8, 11}	{1}, \dots, {6}, {7, \dots, 23}

TABLE 1

column of Table 1. The third column of the table contains the catalogue number of the groups in the primitive groups library of GAP. For instance, the group PSL(3, 2) can be accessed as `PrimitiveGroup( 7, 5 )`. In order to complete the proof of Theorem 2.7 it is necessary to find a partition  $P$  and a subset  $S$  of  $\{1, 2, \dots, n\}$  such that no element in  $SG$  is a transversal for  $P$ . Such subsets and partitions can be found in Table 1.  $\square$

### 5. COMPUTATIONS

In this section we give a brief description of the methods used to perform the various computations which are used above. More specifically, the following were verified computationally using GAP [9].

- (a) In Theorem 1.1, assertion (ii) implies assertion (iii): if  $G$  is one of the groups  $C_5, D_5, \text{AGL}(1, 7), \text{PGL}(2, 7), \text{PSL}(2, 8), \text{PTL}(2, 8)$ , then there exists an  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$  such that  $\langle G, a \rangle \setminus G$  is not idempotent generated.
- (b) In Theorem 1.1, assertion (iii) implies assertion (i): if  $G$  is one of the groups  $\text{AGL}(1, 5), \text{PSL}(2, 5), \text{PGL}(2, 5)$ , then  $\langle G, a \rangle \setminus G$  is idempotent generated for all  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$ .

- (c) One direction in the proof of Theorem 2.7: if  $G$  is any of the groups listed in Theorem 2.7 excluding  $\mathcal{A}_n$  and  $\mathcal{S}_n$ , then  $G$  has the universal transversal property.
- (d) Table 1: if  $G$  is any of the groups listed in Table 1, then  $G$  does not have the universal transversal property.

We prepared a companion webpage [2] that contains full details of these computations. The procedures that were used in these computations were collected into a GAP package, so that the reader can easily reproduce these computations herself. In addition to the GAP package, the webpage contains detailed log files of the computations that were necessary to verify the statements above.

Parts (c) and (d) can be verified using GAP by performing a brute force search. More precisely, if  $G$  is a permutation group of degree  $n$ , then for every subset  $I$  of  $\{1, 2, \dots, n\}$  and for every partition  $P$  of  $\{1, 2, \dots, n\}$  with  $|I|$  classes, we verified that the orbit of  $I$  under  $G$  contained a transversal of  $P$ . These computations are feasible due to the small degrees of the groups under consideration and the efficient methods in GAP for computing with permutation groups.

The idempotents with a specific rank in  $\langle G, a \rangle \setminus G$  can be found using the simple orbit algorithm described in [16] and [17]. Similarly simple orbit calculations, described in the same papers, can be used to test membership in transformation semigroups. As such, the condition of the next lemma, equivalent to  $\langle G, a \rangle \setminus G$  being idempotent generated, can be verified efficiently using GAP.

**Lemma 5.1.** *Let  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$  and let  $E$  denote the set of idempotents of  $\langle G, a \rangle \setminus G$  with rank equal to  $\text{rank } a$ . Then  $\langle G, a \rangle \setminus G$  is idempotent generated if and only if every element of  $GaG$  is contained in  $\langle E \rangle$ .*

*Proof.* Let  $S = \langle G, a \rangle \setminus G$ . Then every element in  $S$  can be given as a product  $g_1 a g_2 a \cdots g_{n-1} a g_n$  for some  $g_1, g_2, \dots, g_n \in G$ . In particular, the set  $GaG$  is a generating set for  $S$ . Thus  $\langle G, a \rangle \setminus G$  is idempotent generated if and only if every element of  $GaG$  lies in the subsemigroup generated by the idempotents of  $S$ . Every element in  $GaG$  has rank equal to that of  $a$ . It follows that  $S$  is idempotent generated if and only if every element of  $GaG$  lies in  $\langle E \rangle$ .  $\square$

If  $G \leq \mathcal{S}_n$  and  $\Gamma$  is a subset of  $\{1, \dots, n\}$ , then the *setwise stabilizer* of  $\Gamma$  in  $G$  is the subgroup  $G_\Gamma = \{g \in G \mid \Gamma g = \Gamma\}$ . The setwise stabilizer induces a group of permutations on the set  $\Gamma$  denoted by  $(G_\Gamma)^\Gamma$ .

**Lemma 5.2.** *Let  $a, b \in \mathcal{T}_n$  where  $\text{rank } b^2 = \text{rank } b$  and such that there exist  $g, h \in G$  with  $(\ker a)g = \ker b$ ,  $(\text{im } a)h = \text{im } b$ , and  $(g^{-1}ah)|_{\text{im } b}(G_{\text{im } b})^{\text{im } b} = b|_{\text{im } b}(G_{\text{im } b})^{\text{im } b}$ . Then  $\langle G, a \rangle = \langle G, b \rangle$ .*

*Proof.* First we verify that the transformation  $g^{-1}ah$  leaves  $\text{im } b$  invariant, and so the expression  $(g^{-1}ah)|_{\text{im } b}(G_{\text{im } b})^{\text{im } b}$  makes sense. Indeed, set  $I = \text{im } b$ . The condition  $\text{rank } b^2 = \text{rank } b$  implies that  $I$  is a transversal for  $\ker b$ , and so  $Ig^{-1}$  is a transversal for  $(\ker b)g^{-1} = \ker a$ . Hence  $Ig^{-1}ah = (\text{im } a)h = \text{im } b$ . Therefore  $(\text{im } b)g^{-1}ah = \text{im } b$  as claimed.

Next we show that the conditions of the lemma imply that there is an element  $u \in G_{\text{im } b}$  such that  $g^{-1}ahu = b$ . As  $(g^{-1}ah)|_{\text{im } b}(G_{\text{im } b})^{\text{im } b} = b|_{\text{im } b}(G_{\text{im } b})^{\text{im } b}$ , we obtain that there is an element  $u \in G_{\text{im } b}$  such that  $g^{-1}ahu$  induces the same permutation on  $\text{im } b$  as  $b$ . Let  $\alpha \in \{1, \dots, n\}$ . Since  $I$  is a transversal for  $\ker b$ , there is an element  $\bar{\alpha} \in I$  such that  $\alpha$  and  $\bar{\alpha}$  are in the same block of  $\ker b$ ; that is

$ab = \bar{a}b$ . This implies that  $\alpha g^{-1}$  and  $\bar{\alpha}g^{-1}$  are in the same block of  $\ker a$ , and so  $\alpha g^{-1}a = \bar{\alpha}g^{-1}a$ . Thus

$$\alpha g^{-1}ahu = \bar{\alpha}g^{-1}ahu = \bar{\alpha}b = \alpha b.$$

Hence the transformations  $g^{-1}ahu$  and  $b$  coincide as claimed. This gives that  $a \in \langle G, b \rangle \setminus G$  and  $b \in \langle G, a \rangle \setminus G$ .  $\square$

Let  $G$  be a subgroup of  $\mathcal{S}_n$ , let  $1 \leq i \leq n-1$ , let  $I_1, I_2, \dots, I_m$  be representatives of orbits of  $G$  on subsets of  $\{1, 2, \dots, n\}$  with size  $i$ , let  $K_1, K_2, \dots, K_r$  be representatives of orbits of  $G$  on the partitions of  $\{1, 2, \dots, n\}$  with  $i$  classes, let  $f_{j,k}$  be an fixed arbitrary element of  $\mathcal{T}_n \setminus \mathcal{S}_n$  with image  $I_j$  and kernel  $K_k$ , let  $T_j$  be a transversal of cosets of the stabilizer  $G_{I_i}$  in  $G$ , and finally, let  $F = \{f_{j,k,t} \mid t \in T_j\}$ . Then to verify that  $\langle G, a \rangle \setminus G$  is idempotent generated for all  $a \in \mathcal{T}_n \setminus \mathcal{S}_n$  with  $\text{rank}(a) = i$ , it suffices, by Lemma 5.2 and the preceding comments, to verify that  $\langle G, a \rangle \setminus G$  is idempotent generated for all  $a \in F$ .

## 6. FINAL REMARKS AND PROBLEMS

We finish the paper by stating some related open problems.

(i) Is it possible to prove the main theorems of this paper without using results that rely on the classification of finite simple groups? In order to give a positive answer to this question, one has to give another proof for Theorem 2.7 that does not use Maróti's bound for the order of a proper primitive group.

(ii) Classify the subgroups  $G$  of  $\mathcal{S}_n$  that together with any singular transformation  $a$  satisfy  $\langle G, a \rangle = \langle a^g \mid g \in G \rangle$ .

(iii) Prove classification theorems analogous to Theorems 1.1 and 1.2 for linear groups and to groups of automorphisms of independence algebras; see [7, 10] for the background theory of independence algebras.

(iv) Classify the pairs  $(G, a)$ , where  $G \leq \mathcal{S}_n$  and  $a \in \mathcal{T}_n$  such that  $\langle G, a \rangle$  is regular. Consider also the corresponding problem for linear groups. As mentioned in the introduction, McAlister proved that for every idempotent  $e \in \mathcal{T}_n$  such that  $\text{rank}(e) = n-1$ , and for all  $G \leq \mathcal{S}_n$  the semigroup  $\langle G, e \rangle$  is regular.

**Acknowledgement.** The research presented in this paper was partially supported by the FCT and FEDER project ISFL-1-143 of Centro de Álgebra da Universidade de Lisboa, and by the FCT and PIDDAC projects PTDC/MAT/69514/2006 and PTDC/MAT/101993/2008. The third author would like to acknowledge the support of the Hungarian Scientific Research Fund (OTKA) grant 72845.

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