Maximal complements in finite groups

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Abstract

Let G be a finite group with a non-abelian minimal normal subgroup N which is a direct product of copies of the simple group X. A parametrisation is given for the conjugacy classes of maximal subgroups of G which complement N in terms of certain homomorphisms taking values in $\operatorname{Aut} X$.

1 Introduction

In [9], the author established that certain iterated wreath products of non-abelian finite simple groups can be generated by two randomly chosen elements with high probability. To do so, it was necessary to understand the number and the indices of maximal subgroups which supplemented the base group of each wreath product. The most difficult of these to control were those maximal subgroups which complement the base group. Indeed Bhattacharjee [3] relies on a very delicate argument to handle such maximal complements in an earlier paper. For the work in [9], the author made use of joint work [7] with Parker where the maximal subgroups of a wreath product which complement its base group were completely classified.

It is the author's hope that probabilistic results such as those in [9] can be produced for far wider classes of groups than merely iterated wreath products. We note that Jaikin-Zapirain and Pyber [5] have recently made important progress on the probabilistic generation of profinite groups. To continue with the methods of [3, 8, 9], however, we shall need results that control the behaviour of maximal subgroups in (finite) groups which are as general as possible. Consequently in this paper we demonstrate how the

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work in [7] may be generalised so as to produce our main results, Theorem 3.4 and its corollary. These are more technical due to the more general situation being considered and require the development of some notation before they can be stated. Nevertheless we are able to extend the methods of [7] so as to obtain a parametrisation which both retains the flavour of the result in [7] and also seems suitable for application in a probabilistic generation setting.

Of course the study of maximal subgroups of finite groups has a rich history and two significant milestones are the work of Aschbacher and Scott [1] and that of Kovács [6]. Although there obviously are links between our work and these earlier papers and also the related paper of Baddeley [2], the author has been unable to find within their far more technical and far-reaching studies the specific parametrisation he requires for the application he has in mind. Consequently it seems both necessary and appropriate to revisit this frequently explored territory.

Around the time that the first draft of this work was completed, Cannon and Holt [4] was published. They return to Kovács's work and describe a practical algorithm for computing representatives for the conjugacy classes of maximal subgroups in a finite group. Our principal construction (see Definition 2.3 below) also appears in [4, Section 2.2] but is used to construct the 'diagonal-type' subgroups rather than complements. Instead the subgroups we are considering are addressed in [4, Section 3.6]. The methods there appear to be different to ours and, although Condition (b) from our Theorem 3.4 is present, an alternative condition appears instead of our Condition (a). Cannon and Holt's condition appears to be a recursive condition to be checked throughout an algorithmic process. Instead our Condition (a) depends on the Classification of Finite Simple Groups (see Lemma 3.2) and is sufficient to guarantee the complement we construct is maximal.

2 Complements and induced homomorphisms

We shall need to consider a number of wreath products, all of which have the same symmetric group as the top group. We set up a few abbreviations to simplify our notation. Fix the set Ω and let L be any group. We write L^Ω for the direct product of copies of L indexed by Ω:

$$L^\Omega = \prod_{\alpha \in \Omega} L_\alpha.$$

The symmetric group $\mathrm{Sym}(\Omega)$ acts on L^Ω by permuting the factors: $L_\alpha^\pi = L_{\alpha\pi}$ for all $\alpha \in \Omega$ and $\pi \in \mathrm{Sym}(\Omega)$. The *wreath product* $L \,\mathrm{wr}_\Omega\, \mathrm{Sym}(\Omega)$ is then the semidirect product of L^Ω by $\mathrm{Sym}(\Omega)$ via this action. We call L^Ω the *base group* and use $W(L)$ to denote this wreath product. We also follow the convention from [7] of writing (x_α) for the element $(x_\alpha)_{\alpha \in \Omega}$ of L^Ω. We always index the entries of such elements by the variable α which ranges over Ω. Given $w = (x_\alpha)\pi$ in $W(L)$, we refer to x_α as the *α-component* of w.

Let N be a normal subgroup of the finite group G and let $Q = G/N$. We may define a map $\chi\colon Q \to \operatorname{Out} N$ by $Ng \mapsto (\operatorname{Inn} N)\sigma_g$, where σ_g denotes the automorphism of N induced by conjugation by the element g from G. This map χ is a homomorphism and is known as a *coupling*. If $x \in Q$, then as $x\chi$ is a coset of $\operatorname{Inn} N$, it makes sense to define

$$E = \{\, (x,\sigma) \in Q \times \operatorname{Aut} N \mid \sigma \in x\chi \,\}.$$

There is a homomorphism from G to $Q \times \operatorname{Aut} N$ given by $g \mapsto (Ng, \sigma_g)$. The image of this homomorphism is E while the kernel is the centre $\mathrm{Z}(N)$ of N. We shall henceforth assume that $\mathrm{Z}(N) = \mathbf{1}$ so that we have an isomorphism from G to E.

Under this isomorphism, the normal subgroup N corresponds to the normal subgroup $\mathbf{1} \times \operatorname{Inn} N$ of E. A complement H to N in G corresponds to a subgroup $\tilde{H}$ of E such that for each $x \in Q$ there exists a unique automorphism σ of N with $(x,\sigma) \in \tilde{H}$. (For if $(x,\sigma_1),(x,\sigma_2) \in \tilde{H}$, then $(1,\sigma_1^{-1}\sigma_2) \in \tilde{H}$, so $\sigma_1^{-1}\sigma_2 \in 1\chi = \operatorname{Inn} N$. Since H complements N, this forces $\sigma_1 = \sigma_2$.) It follows that H determines a homomorphism $\zeta\colon Q \to \operatorname{Aut} N$ via $(x, x\zeta) \in \tilde{H}$ for each $x \in Q$.

This ζ has a further property which we express in terms of the following notation.

Definition 2.1 If $\eta\colon Q \to \operatorname{Aut} L$ is a homomorphism taking values in the automorphism group of some group L, write $\bar{\eta}\colon Q \to \operatorname{Out} L$ for the homomorphism obtained by composing η with the natural map $\operatorname{Aut} L \to \operatorname{Out} L$.

If H is a complement to N then, since $\tilde{H} \leqslant E$, the homomorphism $\zeta\colon Q \to \operatorname{Aut} N$ determined by $\tilde{H}$ must satisfy $\bar{\zeta} = \chi$. Conversely, given a homomorphism $\zeta\colon Q \to \operatorname{Aut} N$ such that $\bar{\zeta} = \chi$, we determine a complement $\{\, (x, x\zeta) \mid x \in Q \,\}$ to $\mathbf{1} \times \operatorname{Inn} N$ in E and hence a complement to N in G. Thus, complements to N in G are in one-one correspondence with homomorphisms $\zeta\colon Q \to \operatorname{Aut} N$ with the property $\bar{\zeta} = \chi$. We refer to this condition $\bar{\zeta} = \chi$ as the *compatibility condition* for complements.

We now assume that N is a non-abelian minimal normal subgroup of G, so that N is a direct product of copies of a non-abelian finite simple group X, say $N = X^{\Omega}$ where $|\Omega| = k$. Then

$$\operatorname{Aut} N \cong W(\operatorname{Aut} X) = (\operatorname{Aut} X) \operatorname{wr}_{\Omega} \operatorname{Sym}(\Omega)$$

and

$$\operatorname{Out} N \cong W(\operatorname{Out} X) = (\operatorname{Out} X) \operatorname{wr}_{\Omega} \operatorname{Sym}(\Omega).$$

Homomorphisms taking values in $\operatorname{Aut} N$ will give rise to permutation representations on Ω. To exploit this we make a further notational definition.

Definition 2.2 If $\eta\colon Q \to W(L)$ is a homomorphism taking values in the wreath product $W(L)$ for some group L, write $\eta^*\colon Q \to \mathrm{Sym}(\Omega)$ for the homomorphism obtained by composing η with the natural map $W(L) \to \mathrm{Sym}(\Omega)$.

Upon comparing Definitions 2.1 and 2.2, we note that given $\eta\colon Q \to W(\mathrm{Aut}\,X)$, then $\eta^* = (\bar{\eta})^*$.

The coupling $\chi\colon Q \to W(\mathrm{Out}\,X)$ associated to the extension G of N by Q now determines the permutation representation $\chi^*\colon Q \to \mathrm{Sym}(\Omega)$. To maintain a permanent record we write $\rho = \chi^*$, since this is determined by G and so fixed throughout our discussion. We now have an action of Q on Ω and we shall write αx for the image of α (from Ω) under the permutation $x\rho$.

The action of G on N permutes the direct factors of N and so determines an action of G on the set Ω. It is straightforward to check the associated permutation representation $G \to \mathrm{Sym}(\Omega)$ equals the composition of the natural map $G \to Q$ with ρ. Since N is a minimal normal subgroup, its direct factors are permuted transitively and hence ρ is a transitive permutation representation. Fix $\omega \in \Omega$, write $\mathrm{Stab}_Q(\omega)$ for the stabiliser of ω in Q, and let $T = \{\, t_\alpha \mid \alpha \in \Omega \,\}$ be a transversal to $\mathrm{Stab}_Q(\omega)$ in Q such that $\omega t_\alpha = \alpha$ for all $\alpha \in \Omega$. We assume $t_\omega = 1$. For $x \in Q$, define $h_{\alpha,x}$ in $\mathrm{Stab}_Q(\omega)$ by

$$t_\alpha x = h_{\alpha,x} t_{\alpha x}.$$

Definition 2.3 Let L be any group and let $\phi\colon \mathrm{Stab}_Q(\omega) \to L$ be a homomorphism. Define $\Phi\colon Q \to W(L)$ by

$$x\Phi = (h_{\alpha,x}\phi)(x\rho).$$

We shall say that this map Φ is *induced* from ϕ.

In this definition the α-component of $x\Phi \in W(L)$ is the image of $h_{\alpha,x}$ under ϕ. The terminology 'induced' seems appropriate due to similarities to such constructions as induced representations. (Cannon and Holt term the similar construction appearing in [4, Definition 2.2] the *wreath monomorphism*.) An identical argument to that used for [7, Lemma 2.1] gives:

Lemma 2.4 *The induced map Φ is a homomorphism.* □

We shall be mostly considering homomorphisms which are induced with respect to the fixed transversal T. There is, however, one point where we need to consider a different choice of transversal. The following, which is proved by the same method as [7, Lemma 2.2], notes that change of transversal simply corresponds to conjugation of our image.

Lemma 2.5 *For each $\alpha \in \Omega$, let k_α be an element of $\mathrm{Stab}_Q(\omega)$ and $t'_\alpha = k_\alpha t_\alpha$. Let $\Phi'\colon Q \to W(L)$ be the homomorphism induced from ϕ with respect to the transversal $T' = \{\, t'_\alpha \mid \alpha \in \Omega \,\}$. Then $\Phi' = \Phi\kappa$ where κ denotes conjugation by the element $(k_\alpha\phi)^{-1}$ of L^Ω.* □

Now consider any homomorphism $\eta\colon Q \to W(L)$ with the property that η^* coincides with the permutation representation ρ. If we consider the restriction of η to the stabiliser $\mathrm{Stab}_Q(\omega)$, then we note that its image $\mathrm{Stab}_Q(\omega)\eta$ must normalise the direct factor L_ω of the base group and hence

$$\mathrm{Stab}_Q(\omega)\eta \leqslant L_\omega \times \bigl(L \,\mathrm{wr}_{\Omega\setminus\{\omega\}}\, \mathrm{Sym}(\Omega \setminus \{\omega\})\bigr).$$

We may therefore make the following definition.

Definition 2.6 If $\eta\colon Q \to W(L)$ is a homomorphism satisfying $\eta^* = \rho$, write $\theta_\eta\colon \mathrm{Stab}_Q(\omega) \to L$ for the homomorphism obtained by composing the restriction of η to the stabiliser $\mathrm{Stab}_Q(\omega)$ with the projection

$$L_\omega \times \bigl(L \,\mathrm{wr}_{\Omega\setminus\{\omega\}}\, \mathrm{Sym}(\Omega \setminus \{\omega\}\bigr) \to L$$

onto the direct factor of the base group indexed by ω.

Thus, $x\theta_\eta$ equals the ω-component of $x\eta$ for each $x \in \mathrm{Stab}_Q(\omega)$.

In particular, if we apply Definition 2.6 when η is an induced homomorphism $\Phi\colon Q \to W(L)$, then, as we chose $t_\omega = 1$, we calculate $x\theta_\Phi = h_{\omega,x}\phi = x\phi$. Hence $\theta_\Phi = \phi$ and so the recipe of Definition 2.6 simply recovers the homomorphism ϕ used to construct Φ.

We could equally well apply Definition 2.6 to the coupling $\chi\colon Q \to W(\mathrm{Out}\,X)$ to yield $\theta_\chi\colon \mathrm{Stab}_Q(\omega) \to \mathrm{Out}\,X$. This map will be important in the context of our compatibility condition $\bar{\zeta} = \chi$, so we record θ_χ as $\psi\colon \mathrm{Stab}_Q(\omega) \to \mathrm{Out}\,X$.

Now suppose that $\zeta\colon Q \to W(\mathrm{Aut}\,X)$ is a homomorphism corresponding to a complement to N in G. This mean, of course, that $\bar{\zeta} = \chi$. Consequently $\zeta^* = (\bar{\zeta})^* = \chi^* = \rho$ and we may use Definition 2.6 to construct $\theta = \theta_\zeta\colon \mathrm{Stab}_Q(\omega) \to \mathrm{Aut}\,X$. Via Definition 2.3, we then induce $\Theta\colon Q \to W(\mathrm{Aut}\,X)$. For each $\gamma \in \Omega$, write $t_\gamma\zeta = (u_\alpha^{(\gamma)})(t_\gamma\rho)$. If $x \in Q$, write $x\zeta = (x_\alpha)(x\rho)$. Then

$$\begin{aligned} h_{\gamma,x}\zeta = (t_\gamma x t_{\gamma x}^{-1})\zeta &= (u_\alpha^{(\gamma)})(t_\gamma\rho)\cdot(x_\alpha)(x\rho)\cdot(t_{\gamma x}^{-1}\rho)(u_\alpha^{(\gamma x)})^{-1} \\ &= \Bigl(u_\alpha^{(\gamma)} x_{\alpha t_\gamma} (u_{\alpha h_{\gamma,x}}^{(\gamma x)})^{-1}\Bigr)(h_{\gamma,x}\rho), \end{aligned}$$

so

$$h_{\gamma,x}\theta = u_\omega^{(\gamma)} x_{\omega t_\gamma} (u_{\omega h_{\gamma,x}}^{(\gamma x)})^{-1} = u_\omega^{(\gamma)} x_\gamma (u_\omega^{(\gamma x)})^{-1}.$$

Write $a_\alpha = u_\omega^{(\alpha)}$. Our calculation shows

$$x\Theta = (a_\alpha x_\alpha a_{\alpha x}^{-1})(x\rho) = (a_\alpha)\cdot(x_\alpha)(x\rho)\cdot(a_\alpha)^{-1}$$

for all $x \in Q$ and we deduce

$$x\zeta = (x\Theta)^a$$

where $a = (a_\alpha) \in (\operatorname{Aut} X)^\Omega$. Thus $\zeta = \Theta\tau_a$, where τ_a is the inner automorphism of $W(\operatorname{Aut} X)$ obtained by conjugating by the element a. Note $a_\omega = u_\omega^{(\omega)} = 1$, since we chose $t_\omega = 1$. Let b denote the image of a under the natural map from $(\operatorname{Aut} X)^\Omega$ onto $(\operatorname{Out} X)^\Omega$. Thus the α-component of b is the ω-component of $t_\alpha\chi$ and so is determined by the group G (and our fixed choice of transversal T) but is independent of the homomorphism ζ and its associated complement.

Conversely, suppose $\phi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$ is a homomorphism satisfying $\bar{\phi} = \psi$. Then the induced map $\Phi\colon Q \to W(\operatorname{Aut} X)$ satisfies $\bar{\Phi} = \Psi$, the latter being the map induced from ψ. The same calculation as above (applied to χ instead of ζ) shows that $\chi = \Psi\tau_b$ for the element $b \in (\operatorname{Out} X)^\Omega$ specified above. Choose $a \in (\operatorname{Aut} X)^\Omega$ whose image in $(\operatorname{Out} X)^\Omega$ equals b and define $\zeta = \Phi\tau_a$. Then by construction $\bar{\zeta} = \bar{\Phi}\tau_b = \chi$ and hence this ζ corresponds to a complement to N in G.

This gives us the following result.

Theorem 2.7 *There exists $b \in (\operatorname{Out} X)^\Omega$ such that*

(i) *every complement to the minimal normal subgroup N in G corresponds to a homomorphism $\zeta\colon Q \to W(\operatorname{Aut} X)$ of the form $\zeta = \Phi\tau_a$, where Φ is induced from a homomorphism $\phi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$ satisfying $\bar{\phi} = \psi$ and where $a = (a_\alpha) \in (\operatorname{Aut} X)^\Omega$ satisfies $a_\omega = 1$ and its image $\bar{a}$ in $(\operatorname{Out} X)^\Omega$ equals b;*

(ii) *given a homomorphism $\phi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$ satisfying $\bar{\phi} = \psi$, there exists $a = (a_\alpha) \in (\operatorname{Aut} X)^\Omega$ with $a_\omega = 1$ and $\bar{a} = b$ such that $\zeta = \Phi\tau_a$ satisfies $\bar{\zeta} = \chi$ and therefore corresponds to a complement to N in G.* □

We shall speak of a complement H to N in G being *associated* to a homomorphism $\phi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$. This indicates that $\tilde{H} = \{\, (x, x\zeta) \mid x \in Q \,\}$ where $\zeta = \Phi\tau_a$ for some a as described in Theorem 2.7.

Proposition 2.8 *Two complements H_1 and H_2 to N in G, associated to the homomorphisms ϕ_1 and ϕ_2 respectively, are conjugate in G if and only if $\phi_1 = \phi_2\sigma$ for some automorphism σ of $\operatorname{Aut} X$ induced by conjugation by an element of $\operatorname{Inn} X$.*

PROOF: The subgroup H_i corresponds to $\tilde{H}_i = \{\, (x, x\zeta_i) \mid x \in Q \,\}$. Here $\zeta_i = \Phi_i\tau_{a_i}$, where Φ_i is induced from $\phi_i\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$ and where $a_i = (a_\alpha^{(i)})$ satisfies $a_\omega^{(i)} = 1$ and $\bar{a}_i = b$ (for the fixed b given by Theorem 2.7). We first assume that $\tilde{H}_1$ and $\tilde{H}_2$ are conjugate in E. Since they complement the subgroup $\mathbf{1} \times \operatorname{Inn} N$, we see this conjugation may be achieved by an element in the latter normal subgroup. Hence $\zeta_1 = \zeta_2\tau_c$ for some $c = (c_\alpha)$

in $\operatorname{Inn} N = (\operatorname{Inn} X)^\Omega$. Since $a_\omega^{(1)} = a_\omega^{(2)} = 1$, we calculate that

$$\theta_{\zeta_1} = \theta_{\Phi_1 \tau_{a_1}} = \theta_{\Phi_1} = \phi_1$$

and that

$$\theta_{\zeta_2 \tau_c} = c_\omega^{-1} \theta_{\Phi_2} c_\omega = c_\omega^{-1} \phi_2 c_\omega.$$

Hence $\phi_1 = \phi_2 \sigma$, where σ denotes conjugation by the element c_ω from $\operatorname{Inn} X$.

Conversely suppose that $\phi_1 = \phi_2 \sigma$ where σ denotes conjugation by some element c_0 from $\operatorname{Inn} X$. Then, for $x \in Q$,

$$x\Phi_1 = \big((h_{\alpha,x}\phi_2)^{c_0} \big)(x\rho) = c^{-1} \cdot (h_{\alpha,x}\phi_2)(x\rho) \cdot c = x\Phi_2 \tau_c$$

where $c = (c_0)_{\alpha \in \Omega} \in \operatorname{Inn} N$. Therefore

$$\zeta_1 = \Phi_2 \tau_c \tau_{a_1} = \zeta_2 \tau_{a_2^{-1} c a_1}.$$

It follows that $\tilde{H}_1 = \tilde{H}_2^{(1, a_2^{-1} c a_1)}$. Now we have $\bar{a}_1 = \bar{a}_2$ (that is, $a_1 \equiv a_2 \pmod{\operatorname{Inn} N}$) and $c \in \operatorname{Inn} N$. Hence $a_2^{-1} c a_1 \in \operatorname{Inn} N$ and we see that $\tilde{H}_1$ and $\tilde{H}_2$ are conjugate in E. Therefore H_1 and H_2 are conjugate in G. □

Although we require the choice of the element a appearing in Theorem 2.7 to fully specify a complement to N, we may use just the homomorphisms $\phi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$ to parametrise the conjugacy classes of such complements. Let us define an equivalence relation $\sim$ on the set of such functions by $\phi_1 \sim \phi_2$ if and only if $\phi_1 = \phi_2 \sigma$ for some automorphism σ induced by conjugation by an element of $\operatorname{Inn} X$. We then deduce:

Corollary 2.9 *The conjugacy classes of complement to N in G are in one-one correspondence with equivalence classes under $\sim$ of homomorphisms $\phi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$ satisfying the compatibility condition $\bar{\phi} = \psi$.* □

3 Conditions for maximality

We maintain the notation of the previous section. In particular, let H be a complement to the non-abelian minimal normal subgroup N in G and write $\tilde{H} = \{ (x, x\zeta) \mid x \in Q \}$, where the homomorphism $\zeta\colon Q \to W(\operatorname{Aut} X)$ satisfies $\bar{\zeta} = \chi$. By Theorem 2.7(i), we may write $\zeta = \Phi\tau_a$, where $a \in (\operatorname{Aut} X)^\Omega$ and Φ is induced from $\phi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$ with $\bar{\phi} = \psi$.

It is straightforward to see that H is a maximal subgroup of G if and only if it does not normalise a non-trivial proper subgroup of N. Passing to E and its subgroup $\tilde{H}$, we note that $\tilde{H}$ normalises a subgroup $\mathbf{1} \times M$ of $\mathbf{1} \times \operatorname{Inn} N$ if and only if the image of ζ normalises M and, since $\zeta = \Phi\tau_a$, this is in turn equivalent to the image of Φ normalising the subgroup $M^{a^{-1}}$ of $\operatorname{Inn} N$. We record this observation as a lemma.

Lemma 3.1 *The complement H corresponding to $\zeta = \Phi\tau_a$ is maximal in G if and only if the image of Φ normalises no non-trivial proper subgroup of* $\operatorname{Inn} N$. $\square$

In particular, when determining the maximality of the complement corresponding to $\phi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$, we need only consider the induced homomorphism Φ.

Let us first assume that our complement is a maximal subgroup of G; that is, that the image of Φ in $W(\operatorname{Aut} X)$ normalises no non-trivial proper subgroup of $\operatorname{Inn} N$.

Lemma 3.2 *If the image of Φ normalises no non-trivial proper subgroup of* $\operatorname{Inn} N$, *then the image of ϕ in* $\operatorname{Aut} X$ *contains* $\operatorname{Inn} X$.

PROOF: First suppose that the image of ϕ normalises a non-trivial proper subgroup R of $\operatorname{Inn} X$. It follows that the image of the stabiliser $\operatorname{Stab}_Q(\omega)$ under Φ normalises the subgroup

$$R_\omega = \{\, (x_\alpha) \in (\operatorname{Aut} X)^\Omega \mid x_\omega \in R,\ x_\alpha = 1 \text{ for } \alpha \neq \omega \,\}$$

of $\operatorname{Inn} N$. The distinct conjugates of R_ω under the action of $Q\Phi$ lie in distinct factors of $\operatorname{Inn} N$. Therefore the subgroup S generated by these conjugates is a direct product

$$S = \prod_{\alpha \in \Omega} R_\alpha$$

of copies R_α of R. This subgroup S is then a non-trivial proper subgroup of $\operatorname{Inn} N$ which is normalised by the image of Φ, contrary to assumption. Hence the image of ϕ normalises no non-trivial proper subgroup of $\operatorname{Inn} X$.

Now consider $L = \operatorname{Stab}_Q(\omega)\phi \cap \operatorname{Inn} X$. If $L \neq \mathbf{1}$, then L is a normal subgroup of $\operatorname{Stab}_Q(\omega)\phi$ and taking $R = L$ in the previous paragraph shows that $L = \operatorname{Inn} X$; that is, the image of ϕ contains $\operatorname{Inn} X$ and the lemma would be proved.

The only remaining possibility is that $L = \mathbf{1}$. An argument of Wilson (see the proof of [10, Lemma 2.1]), which depends on the Classification of Finite Simple Groups, then shows that the centraliser in $\operatorname{Inn} X$ of a minimal normal subgroup M of $\operatorname{Stab}_Q(\omega)\phi$ is non-trivial. This centraliser certainly is a proper subgroup of $\operatorname{Inn} X$ and it is normalised by $\operatorname{Stab}_Q(\omega)\phi$, so we could take $R = \mathrm{C}_{\operatorname{Inn} X}(M)$ in the first paragraph. Thus, it is not possible that $L = \mathbf{1}$ and the proof of the lemma is complete. $\square$

Lemma 3.3 *Suppose that the image of Φ normalises no non-trivial proper subgroup of* $\operatorname{Inn} N$ *and that ϕ is equal to the restriction of some homomorphism $\hat{\phi}\colon H \to$* $\operatorname{Aut} X$, *where H is a subgroup of Q containing* $\operatorname{Stab}_Q(\omega)$, *then $H =$* $\operatorname{Stab}_Q(\omega)$.

PROOF: The subgroup H yields a (possibly trivial or improper) block system

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_s$$

for the action of Q on Ω. Assume that $\omega \in \Omega_1$, so that $\Omega_1 = \omega H$, the orbit of H containing ω. Since $\mathrm{Stab}_Q(\omega)$ also equals the stabiliser of ω in H, there exists a transversal $U = \{ u_\alpha \mid \alpha \in \Omega_1 \}$ to $\mathrm{Stab}_Q(\omega)$ in H such that $\omega u_\alpha = \alpha$ for each $\alpha \in \Omega_1$ and we may assume that $u_\omega = 1$. Let $V = \{v_1, v_2, \dots, v_s\}$ be a transversal to H in Q such that $\Omega_1 v_i = \Omega_i$ for $i = 1, 2, \dots, s$ and assume that $v_1 = 1$. Then $T' = \{ u_\alpha v_i \mid \alpha \in \Omega_1,\ 1 \leqslant i \leqslant s \}$ is a transversal to $\mathrm{Stab}_Q(\omega)$ in Q. Write t'_α for the element of T' satisfying $\omega t'_\alpha = \alpha$. Then $t'_\alpha = u_\alpha$ for all $\alpha \in \Omega_1$ since $v_1 = 1$. Let $\Phi' \colon Q \to W(\mathrm{Aut}\, X)$ be the homomorphism induced from ϕ with respect to T'. Lemma 2.5 tells us that Φ and Φ' differ by conjugation by some element of $(\mathrm{Aut}\, X)^\Omega$.

For each $\alpha \in \Omega_1$, let $\psi_\alpha = u_\alpha \hat{\phi} \in \mathrm{Aut}\, X$ and define

$$D = \{ (x^{\psi_\alpha^{-1}})_{\alpha \in \Omega_1} \mid x \in \mathrm{Inn}\, X \},$$

a diagonal subgroup of $(\mathrm{Inn}\, X)^{\Omega_1}$. (Here $x^{\psi_\alpha^{-1}}$ denotes the conjugate $\psi_\alpha x \psi_\alpha^{-1}$ of x by ψ_α^{-1}.) We shall view D as being embedded in $(\mathrm{Inn}\, X)^\Omega$ in the obvious way. Let $y \in H$. Then if $\alpha \in \Omega_1$, we have

$$h'_{\alpha,y} = t'_\alpha y (t'_{\alpha y})^{-1} = u_\alpha y u_{\alpha y}^{-1}$$

and so

$$h'_{\alpha,y}\phi = h'_{\alpha,y}\hat{\phi} = \psi_\alpha (y\hat{\phi}) \psi_{\alpha y}^{-1}.$$

Therefore, upon conjugating an element of D by $y\Phi'$, we find

$$\begin{aligned} \Big[(x^{\psi_\alpha^{-1}})_{\alpha\in\Omega_1}\Big]^{y\Phi'} &= \Big[(x^{\psi_\alpha^{-1}})_{\alpha\in\Omega_1}\Big]^{(h'_{\alpha,y}\phi)_{\alpha\in\Omega}(y\rho)} \\ &= \Big[\Big((x^{\psi_\alpha^{-1}})^{h'_{\alpha,y}\phi}\Big)_{\alpha\in\Omega_1}\Big]^{y\rho} = \Big[(z^{\psi_{\alpha y}^{-1}})_{\alpha\in\Omega_1}\Big]^{y\rho} = (z^{\psi_\alpha^{-1}})_{\alpha\in\Omega_1}, \end{aligned}$$

where $z = x^{y\hat{\phi}} \in \mathrm{Inn}\, X$. In particular, D is normalised by $H\Phi'$. Conjugating by the appropriate element of $(\mathrm{Aut}\, X)^\Omega$, we deduce that $H\Phi$ normalises some diagonal subgroup E of $(\mathrm{Inn}\, X)^{\Omega_1}$.

Note that $\{v_1\Phi, v_2\Phi, \dots, v_s\Phi\}$ is a transversal to $H\Phi$ in $Q\Phi$ and that $E^{v_i\Phi} \leqslant (\mathrm{Inn}\, X)^{\Omega_i}$ for each i. Hence the conjugates $E^{v_i\Phi}$ generate a direct product

$$F = E^{v_1\Phi} \times E^{v_2\Phi} \times \cdots \times E^{v_s\Phi}$$

and this is normalised by $Q\Phi$. Our assumption that the image of Φ normalises no non-trivial proper subgroup of $\mathrm{Inn}\, N$ then forces $|\Omega_1| = 1$. Therefore $H = \mathrm{Stab}_Q(\omega)$, as required. □

The previous two lemmas tell us that if a complement is maximal then we have two conditions on the associated homomorphism $\phi\colon \mathrm{Stab}_Q(\omega) \to \mathrm{Aut}\, X$. Conversely consider a complement to N in G associated to ϕ and assume that the conditions established above hold; that is,

(a) the image of ϕ in $\mathrm{Aut}\, X$ contains $\mathrm{Inn}\, X$, and

(b) ϕ is not the restriction of some homomorphism $\hat{\phi}\colon J \to \mathrm{Aut}\, X$ where $\mathrm{Stab}_Q(\omega) < J \leqslant Q$.

We seek to show that our complement is maximal in G and, in view of Lemma 3.1, this is equivalent to establishing that the image of Φ does not normalise any non-trivial proper subgroup of $\mathrm{Inn}\, N$.

Let L be a subgroup of $\mathrm{Inn}\, N$ which is normalised by the image of Φ. Let L_α be the subgroup of $\mathrm{Inn}\, X$ which is the image of L under the projection of $\mathrm{Inn}\, N$ onto its direct factor indexed by α. Since the permutation representation ρ used to construct the induced map is transitive and since $Q\Phi$ normalises L, it follows that the subgroups L_α are conjugate in $\mathrm{Aut}\, X$. We therefore have three possibilities:

(i) $L_\alpha = \mathrm{Inn}\, X$ for all $\alpha \in \Omega$,

(ii) $\mathbf{1} < L_\alpha < \mathrm{Inn}\, X$ for all $\alpha \in \Omega$, or

(iii) $L_\alpha = \mathbf{1}$ for all $\alpha \in \Omega$.

Of course, the third case implies $L = \mathbf{1}$, which is perfectly acceptable for a subgroup of $\mathrm{Inn}\, N$ normalised by $Q\Phi$. We shall show that our two assumptions (a) and (b) will force $L = \mathrm{Inn}\, N$ in Case (i) and will yield a contradiction in Case (ii).

We first deal with Case (ii). Note that L is also normalised by the image of $\mathrm{Stab}_Q(\omega)$ under Φ and hence L_ω is normalised by the image of $\mathrm{Stab}_Q(\omega)\Phi$ under the projection onto the direct factor of the base group of $\mathrm{Aut}\, N$ indexed by ω. However, the image of $\mathrm{Stab}_Q(\omega)\Phi$ under this projection is merely the image of ϕ and thus L_ω is normalised by the image of ϕ. Our first assumption (a) then shows that L_ω is a normal subgroup of $\mathrm{Inn}\, X$ and since $X (\cong \mathrm{Inn}\, X)$ is simple we obtain the required contradiction.

We then turn to Case (i) where $L_\alpha = \mathrm{Inn}\, X$ for all $\alpha \in \Omega$. This says that L is a subdirect product in $\mathrm{Inn}\, N$. Since $\mathrm{Inn}\, X$ is isomorphic to the non-abelian simple group X, this subdirect product is a direct product of diagonal subgroups, say

$$L = D_1 \times D_2 \times \cdots \times D_s$$

where $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_s$ is a partition of Ω and each D_i is a diagonal subgroup of the direct product $(\mathrm{Inn}\, X)^{\Omega_i}$. These D_i are then minimal normal subgroups of L, so conjugation by elements of $Q\Phi$ permutes them. Since the action of Q on Ω is determined by $\rho = \Phi^*$, we see that $\{\Omega_1, \Omega_2, \ldots, \Omega_s\}$ is a block system for Q on Ω. Assume that $\omega \in \Omega_1$ and let J be the subgroup of Q which stabilises the block Ω_1. Then $\mathrm{Stab}_Q(\omega) \leqslant J$ and $\Omega_1 = \omega J$, the orbit of ω under the action of J.

Now $J\Phi$ normalises the subgroup $(\mathrm{Inn}\, X)^{\Omega_1}$ of $\mathrm{Inn}\, N$ and hence normalises $L \cap (\mathrm{Inn}\, X)^{\Omega_1} = D_1$. We may write

$$D_1 = \{ (\sigma_{x\phi_\alpha})_{\alpha \in \Omega_1} \mid x \in X \}$$

where each ϕ_α is an automorphism of X and σ_g denotes the inner automorphism of X obtained by conjugating by g. Without loss of generality we shall assume that ϕ_ω is the identity map. Then

$$(\sigma_{x\phi_\alpha})_{\alpha \in \Omega_1} \mapsto x$$

is an isomorphism. The action of $J\Phi$ on D_1 then induces an action of J on X via this isomorphism and hence produces a homomorphism $\hat{\phi}\colon J \to \mathrm{Aut}\, X$.

Consider the restriction of $\hat{\phi}$ to $\mathrm{Stab}_Q(\omega)$. If $y \in \mathrm{Stab}_Q(\omega)$, then the ω-component of

$$[(\sigma_{x\phi_\alpha})_{\alpha \in \Omega_1}]^{y\Phi}$$

is

$$\sigma_x^{h_{\omega,y}\phi} = \sigma_x^{y\phi} = \sigma_{x(y\phi)}.$$

Hence $y\hat{\phi}$ is the map $x \mapsto x(y\phi)$; that is, $y\hat{\phi} = y\phi$. We may now use assumption (b) to deduce that $J = \mathrm{Stab}_Q(\omega)$ and hence $|\Omega_i| = 1$ for all i. Therefore $L = (\mathrm{Inn}\, X)^\Omega = N$ in Case (i).

We now conclude that under our assumptions there is no non-trivial proper subgroup of $\mathrm{Inn}\, N$ normalised by $Q\Phi$ and this tells us that our complement to N is a maximal subgroup of G. This completes the proof of the following theorem.

Theorem 3.4 *Let G be a finite group with minimal normal subgroup N which is a direct product of copies of a non-abelian finite simple group X indexed by the set Ω. Let $Q = G/N$ and let Q act on Ω via the coupling $\chi\colon Q \to W(\mathrm{Aut}\, X)$. A complement to N in G associated to the homomorphism $\phi\colon \mathrm{Stab}_Q(\omega) \to \mathrm{Aut}\, X$ (necessarily satisfying the condition $\bar{\phi} = \psi = \theta_\chi$) is maximal in G if and only if*

(a) *the image of ϕ in $\mathrm{Aut}\, X$ contains $\mathrm{Inn}\, X$, and*

(b) *ϕ is not the restriction of some homomorphism $\hat{\phi}\colon J \to \mathrm{Aut}\, X$ where $\mathrm{Stab}_Q(\omega) < J \leqslant Q$.* □

Passing to conjugacy classes of complements as described in Corollary 2.9 gives us the following parametrisation, which we express in terms of the notation developed earlier.

Corollary 3.5 *The conjugacy classes of complements to N in G which are maximal subgroups of G are in one-one correspondence with equivalence classes of homomorphisms $\phi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$ such that*

(i) $\bar{\phi} = \psi$;

(ii) *the image of ϕ contains $\operatorname{Inn} X$;*

(iii) *ϕ is not the restriction of some homomorphism $\hat{\phi}\colon J \to \operatorname{Aut} X$ where $\operatorname{Stab}_Q(\omega) < J \leqslant Q$;*

under the equivalence relation $\sim$, where $\phi_1 \sim \phi_2$ if and only if $\phi_1 = \phi_2\sigma$ for some automorphism σ of $\operatorname{Aut} X$ induced by conjugation by an element of $\operatorname{Inn} X$. □

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