Combinatorics 1:
The art of counting
Preface

This is the first of a three-part set of lecture notes on Advanced Combinatorics, for the module MT5821 of that title at the University of St Andrews.

Roughly speaking, Combinatorics is the study of arranging objects according to certain rules. The natural questions are: is the arrangement possible? If so, in how many different ways can it be made? What extra properties such as symmetry do these arrangements have? This part of the notes is mainly concerned with the second question: for basic objects such as subsets of a set, there is no doubt about their existence, and we want to know how many there are with various properties. We will touch briefly on the first question in a section on combinatorics of subsets, treating Ramsey’s theorem and Steiner systems.

The most powerful tool in enumerative combinatorics is the use of formal power series, and we spend some time on these objects and their properties.

The syllabus for the module describes the three options as follows:

1. **Enumerative combinatorics:** basic counting, formal power series and their calculus, recurrence relations, $q$-analogues, group action and cycle index, species, asymptotic results.

2. **Graphs, codes and designs:** strongly regular graphs, $t$-designs, optimality for block designs, codes and weight enumerators, matroids and Tutte polynomial, MacWilliams relations.

3. **Projective and polar spaces:** geometry of vector spaces, combinatorics of projective planes, sesquilinear and quadratic forms and their classification, diagram geometry, classical groups.

These notes refer to the first section, delivered for the first time in the second semester of 2013–2014.

Many images in the notes are taken from that great St Andrews resource, the MacTutor History of Mathematics website.

Counting combinatorial objects can mean various different things:

- Best of all is an exact formula, such as the formula $2^n$ for the number of subsets of a set of size $n$. This formula is also easy to evaluate for given $n$, and tells us how fast the number in question grows as a function of $n$. 

- Some formulae are more complicated, such as the following formula for the number of partitions of an \( n \)-set with \( k \) parts:

\[
S(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n.
\]

- Even when there is a simple formula, it may be difficult to estimate its magnitude. The number of permutations of an \( n \)-set is \( n! \), and Stirling’s formula gives us the asymptotic behaviour:

\[
n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.
\]

- Sometimes even an asymptotic formula is too much to ask, and we have to be content with some kind of approximation.

- Maybe we can’t give an exact or asymptotic formula, but we have a recurrence relation. The number \( B(n) \) of partitions of an \( n \)-set satisfies

\[
B(n) = \sum_{i=1}^{n} \binom{n-1}{i-1} B(n-i).
\]

- Sometimes we want to generate all the objects counted by a combinatorial formula. In this case we need either a method of stepping from any object to the next (as the odometer or mileage gauge in a car does, with strings of digits), or a method of computing the \( i \)th object in some ordering directly from \( i \).

The tentative weekly lecture schedule is as follows:

1. Counting subsets: binomial coefficients, identities, generating functions
2. Combinatorics of subsets: Ramsey’s theorem, Steiner triple systems
3. Counting partitions and permutations: Stirling and Bell numbers, factorials
4. Formal power series: operations, etc.
5. An example: Catalan numbers
6. Unimodality
7. q-series and what they count
8. Inclusion-exclusion, incl. chromatic polynomial of a graph
9. Group actions, orbit counting, cycle index
10. Asymptotics; Stirling’s formula

In fact, this may get substantially modified, depending on how the lectures turn out. Lecture notes will be provided after the lectures, and the content of the course is defined by the content of the lectures and the exercises.

Peter J. Cameron
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