The following questions cover the course material so far. I will be happy to look at any solutions you want to hand in.

4.1. Let \( n, k, l \) be positive integers with \( n \geq k \geq l \). Prove that

\[
\binom{n}{k} \binom{k}{l} = \binom{n}{l} \binom{n-l}{k-l}
\]

(a) by using the formula for binomial coefficients;
(b) by a counting argument;
(c) by calculating a term of \( (x + (y + z))^n = ((x+y) + z)^n \).

4.2.

(a) In how many ways can 12 sweets be distributed among 5 children?
(b) In how many ways can this be done if each child is to have at least one sweet?
(c) [Harder!] In how many ways can this be done if each child is to have at least one and at most three sweets?

4.3. By considering \( (1 + \omega)^n \), where \( \omega = (-1 + \sqrt{-3})/2 \) is a cube root of unity, calculate

\[
\sum_{i=0}^{[n/3]} \binom{n}{3i}
\]

(Hint: Separate the congruence classes for \( n \mod 6 \).)

4.4. Prove the following:

\[
\begin{align*}
\text{(a)} & \quad \sum_{i=0}^{k} \binom{n+i}{i} = \binom{n+k+1}{k} \\
\text{(b)} & \quad \sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}.
\end{align*}
\]
4.5. Prove the following about Fibonacci numbers. I take the convention that \( F_0 = F_1 = 1 \). Then \( F_n = F_{n-1} + F_{n-2} \) for \( n \geq 2 \).

(a) The number of compositions (ordered sums) for \( n \) as a sum of 1s and 2s is \( F_n \).

(b) Given \( n \) seats in a line, the number of ways of choosing a subset of them including no two consecutive positions is \( F_{n+1} \). If the \( n \) seats are in a circle (with \( n \geq 3 \), the number is \( F_n + F_{n-2} \).

(c) \( F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1} \).

(d) \( \sum_{k=0}^n F_k = F_{n+2} - 1 \).

(e) \( \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} = F_n \). [Hint: Part (a).]

4.6. Show that the number of ways of covering a \( 2 \times n \) board with dominoes (\( 1 \times 2 \) rectangles) is \( F_n \). [Harder] What about a \( 3 \times n \) board?

4.7. Show that the sequence with \( n \)th term \( a_n = F_n^2 \) satisfies a linear recurrence relation in which the \( n \)th term depends on the three preceding terms for \( n \geq 3 \).

4.8. Let \( X = \mathbb{Z}/(13) \) (the integers mod 13), and let \( B_1 = \{0, 3, 4\} \) and \( B_2 = \{0, 6, 8\} \). Show that every non-zero element \( x \in X \) can be written uniquely as \( x = y - z \) where \( y, z \in B_i \) for some \( i \). Hence show that the 26 translates of \( B_1 \) and \( B_2 \) mod 13 are the blocks of a Steiner system \( S(2, 3, 13) \).

4.9. This exercise shows that \( R_2(3, 3, 3) = 17 \). We have \( R_2(3, 3, 3) \leq 17 \), so we need to construct a colouring of the 2-subsets of a 16-set with three colours and no monochromatic triangles.

Take \( X \) to be the set of all complementary pairs \( \{x, \bar{x}\} \) of binary vectors of length 5. (For example \( \{11010, 00101\} \)). There are \( 2^5/2 = 16 \) pairs. Note that mod-2 addition is well defined on these pairs: if \( x + y = z \), then \( \bar{x} + \bar{y} = \bar{z} \), etc. Join two pairs by a red edge if their sum is a pair of vectors with 1 and 4 ones; a green edge if their sum is a pair of vectors with 2 and 3 ones, and the 2 ones (or zeros) are consecutive in the cyclic order; and by a blue edge otherwise.

For example, the edge from \( \{11010, 00101\} \) to \( \{11110, 00001\} \) is red; from \( \{11010, 00101\} \) to \( \{10110, 01001\} \) is green; and from \( \{11010, 00101\} \) to \( \{01110, 10001\} \) is blue.

Show that there is no monochromatic triangle. (Hint: use symmetry to show that, if there is such a triangle, then there is one containing \( \{00000, 11101\} \).)