6 Projective geometry

First, a brief reminder about projective planes. One of these is a geometry of points and lines, each line being a set of points of size greater than 1, satisfying

- two points lie on a unique line;
- two lines meet in a unique point;
- there exist four points, with no three collinear.

A finite projective plane has an order, an integer $n > 1$ such that

- there are $n^2 + n + 1$ points, and $n^2 + n + 1$ lines;
- each line contains $n + 1$ points, and each point lies on $n + 1$ lines.

We constructed a projective plane by taking a 3-dimensional vector space $V$ over a field $F$ and letting the points and lines of the plane be the 1- and 2-dimensional subspaces of $V$, identifying a line with the set of points it contains.

If the field $F$ is finite, the resulting projective plane is finite, and its order is $|F|$. Now finite fields are described as follows:

**Theorem 6.1** The number of elements in a finite field is a prime power. Conversely, if $q$ is a prime power, there is a field with $q$ elements, unique up to isomorphism, which is denoted by $GF(q)$.

This procedure immediately generalises to higher dimension. Note the dimension shift: just as a projective plane comes from a 3-dimensional vector space, so $n$-dimensional projective space comes from a $(n + 1)$-dimensional vector space!
6.1 Definition

Let $F$ be a field, and $n$ an integer at least 2.

The $n$-dimensional projective geometry $\text{PG}(n,F)$ is obtained as follows. Let $V$ be an $(n+1)$-dimensional vector space over $V$. Then geometric objects are identified with subspaces of $V$:

- **points** are 1-dimensional vector subspaces;
- **lines** are 2-dimensional vector subspaces;
- **planes** are 3-dimensional vector subspaces;
- **solids** are 4-dimensional vector subspaces;
- **...**
- **hyperplanes** are $n$-dimensional vector subspaces.

If $F$ is the finite field $\text{GF}(q)$ with $q$ elements, we call this geometry $\text{PG}(n,q)$ instead of $\text{PG}(n,\text{GF}(q))$.

Properties of vector space dimension immediately give us familiar geometric properties of these objects. For example:

**Proposition 6.2** In an $n$-dimensional projective geometry,

(a) two points lie in a unique line;

(b) three non-collinear points lie in a unique plane;

(c) the points and lines in a plane form a projective plane.

**Proof** (a) Two 1-dimensional subspaces span a unique 2-dimensional subspace. (b) Three 1-dimensional subspaces span a subspace of dimension 2 or 3; if they are not collinear, the dimension must be 3. (c) The points and lines in a plane are the 1- and 2-dimensional subspaces in a 3-dimensional vector space, which form a projective plane by the argument we gave earlier.

The next theorem describes subspaces, quotients, and duality.

**Proposition 6.3** In the geometry $\text{PG}(n,F)$:

(a) Take a fixed $i$-space $U$, and consider the geometry whose points, lines, ... are the points, lines, ... of $U$. This is a projective geometry $\text{PG}(i,q)$.
(b) Take a fixed $i$-space $U$, and consider the geometry whose points, lines, ... are the $(i + 1)$-spaces, $(i + 2)$-spaces, ... containing $U$. This geometry is a projective geometry $\text{PG}(n - i - 1, F)$.

(c) Consider the geometry whose points, lines, ... are the hyperplanes, $(n - 2)$-spaces, .... This is a projective geometry $\text{PG}(n, F)$ isomorphic to the original geometry. [In other words, the projective geometry is self-dual.]

**Proof** The first two results follow from the fact that subspaces and quotients of a vector space are themselves vector spaces. For the third, note that there is a natural bijection between the subspaces of a vector space and its dual, where an $i$-dimensional subspace of $V$ corresponds to its annihilator, a subspace of codimension $i$ in $V^*$.

### 6.2 Finite projective space

Over a finite field, we have the possibility to do some counting.

Since an $n$-dimensional vector space over a field $F$ can be identified with the set of all $n$-tuples of field elements (by choosing a basis and setting up coordinates), we see that, if $|F| = q$, then the number of vectors in the $n$-dimensional vector space is $q^n$.

The important function for counting in projective geometries is the $q$-binomial or Gaussian coefficient. The name recalls the term “binomial coefficient”, of which it is a generalisation, as we will see. It is defined as follows:

\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1)(q^{k-1} - 1) \cdots (q - 1)}.
\]

To see that it is a generalisation of the binomial coefficients, use l’Hôpital’s rule to show that

\[
\lim_{q \to 1} \left[ \begin{array}{c} n \\ k \end{array} \right]_q = \left( \begin{array}{c} n \\ k \end{array} \right).
\]

You can find much more about the Gaussian coefficients in the notes for the first variant of the course, at [http://www-groups.mcs.st-andrews.ac.uk/~pjc/Teaching/MT5821/1/](http://www-groups.mcs.st-andrews.ac.uk/~pjc/Teaching/MT5821/1/)

**Theorem 6.4** The number of $k$-dimensional subspaces of an $n$-dimensional vector space over a field with $q$ elements is \( \left[ \begin{array}{c} n \\ k \end{array} \right]_q \).
**Proof** We can specify a $k$-dimensional subspace by giving a basis for it, a sequence of $k$ linearly independent vectors. The first vector $v_1$ can be any non-zero vector ($q^n - 1$ choices); the second vector $v_2$, anything except for a scalar multiple of $v_1$ ($q^n - q$ choices); the third, anything outside the 2-dimensional span of $v_1$ and $v_2$ ($q^n - q^2$ choices); and so on. So there are altogether

$$(q^n - 1)(q^n - q)(q^n - q^2)\cdots(q^n - q^{k-1})$$

choices for a linearly independent $k$-tuple.

Every $k$-tuple spans a $k$-dimensional subspace, but we need to divide by the number of linearly independent $k$-tuples in a given $k$-dimensional space; this number is obtained from the displayed formula by substituting $k$ for $n$.

Dividing, and cancelling powers of $q$, gives the result.

In agreement with our earlier result, the numbers of points and lines in the projective plane over the field with $q$ elements is

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix}_q = \frac{q^3 - 1}{q - 1} = q^2 + q + 1, \quad \begin{pmatrix} 3 \\ 2 \end{pmatrix}_q = \frac{(q^3 - 1)(q^2 - 1)}{(q^2 - 1)(q - 1)} = q^2 + q + 1.$$

Recall that in any calculation of this sort, there is a difference of one between the vector space dimension and the geometric dimension!

### 6.3 Steiner systems and designs

Recall that a Steiner system $S(t, k, n)$ consists of a set of $n$ points and a collection of $k$-element sets of points called blocks, with the property that any $t$ points lie in a unique block.

**Proposition 6.5** The points and lines of $\text{PG}(n, q)$ form a Steiner system $S(2, q + 1, (q^{n+1} - 1)/(q - 1))$.

**Proof** There are $\begin{pmatrix} n+1 \\ 1 \end{pmatrix}_q = (q^{n+1} - 1)/(q - 1)$ points. Any line has $\begin{pmatrix} 2 \\ 1 \end{pmatrix}_q = q + 1$ points. We have seen that two points lie in a unique line.

In the case $n = 2$ (projective planes), we get $q^2 + q + 1$ points, as expected.

Other dimensions of subspaces give rise to more general structures. We will not spend very much time on these.

A $t$-$(v, k, \lambda)$ design consists of a set of $v$ points, with a collection of $k$-element sets of points called blocks, with the property that any set of $t$ points is contained
in exactly $\lambda$ blocks. So a Steiner system $S(t, k, n)$ is a $t$-$(n, k, 1)$ design. The term “design” and the use of the letter $v$ comes from the design of experiments in statistics, where the points are “varieties” to be tested in an experiment. Statisticians call a 2-$(v, k, \lambda)$ design a balanced incomplete-block design, and are not very much interested in designs with $t > 2$.

We need one small counting result for $t = 2$.

**Theorem 6.6** In a 2-$(v, k, \lambda)$ design,

(a) the number of blocks containing a point $p$ is $r = (v-1)\lambda / (k-1)$, independent of $p$;

(b) the total number of blocks is $b = v(v-1)\lambda / k(k-1)$.

**Proof** (a) Given $p$, count pairs $(B, p')$ where $B$ is a block containing $p$ and $p'$ another point of $B$. There are $r(k-1)$ choices (counting $B$ first and then $p'$, where $r$ is the number of blocks containing $p$). But counting $p'$ first and then $B$, there are $(v-1)\lambda$ such pairs. So $r = (v-1)\lambda / (k-1)$, and does not depend on $p$.

(b) Count pairs $(p, B)$ where $p$ is a point and $B$ a block containing it. Choosing $p$ first, there are $vr$ choices; choosing $B$ first, there are $bk$ choices, where $b$ is the number of blocks.

The notation $r$ for the number of blocks containing a point also comes from the statistical applications, where $r$ stands for replication number.

A design can be described by an incidence matrix $M$, whose rows are indexed by the points and columns by blocks; the $(p, B)$ entry is 1 if $p \in B$, 0 otherwise. This matrix is $v \times b$. Let $j$ denote a $b \times 1$ column vector, and $j^*$ a $1 \times v$ row vector.

**Theorem 6.7** If $M$ is the incidence matrix of a 2-$(v, k, \lambda)$ design, then

$$Mj = kj, \quad j^*M = rM, \quad MM^\top = (r - \lambda)I + \lambda J.$$

**Proof** The first equation counts ones in a column of $M$, that is, points in a block; the second counts ones in a row, that is, blocks containing a point; and the third counts common ones in a pair of rows, which is $r$ if the two rows are equal and $k$ otherwise.

Now we turn back to projective spaces and construct some 2-designs.
Theorem 6.8 Suppose that $2 \leq m \leq n - 1$. Then the points and $m$-dimensional subspaces in $\text{PG}(n,q)$ form a $2-(v,k,\lambda)$ design, where $v = \begin{bmatrix} n+1 \end{bmatrix}_q$, $k = \begin{bmatrix} m+1 \end{bmatrix}_q$, and $\lambda = \begin{bmatrix} n-1 \end{bmatrix}_q$.

Proof The calculation of the number of points, and the number of points in a block, are as before.

Given two distinct points $p, p'$, how many blocks contain both? We know that $p$ and $p'$ are 1-dimensional subspaces of an $(n + 1)$-dimensional space $V$, and so they span a 2-dimensional subspace $W$. By Proposition 6.3(b), the $(m + 1)$-dimensional subspaces of $V$ containing $W$ is equal to the number of $(m-1)$-dimensional subspaces of the $(n-1)$-dimensional quotient space $V/W$, and so is $\begin{bmatrix} n-1 \end{bmatrix}_q$, as claimed.

Example The geometry $\text{PG}(3,2)$ has $2^4 - 1 = 15$ points. The lines are the blocks of a $2-(15,3,1)$ design (a Steiner triple system of order 15), while the planes are the blocks of a $2-(15,7,3)$ design.

The number of blocks of the latter design is

$$\begin{bmatrix} 4 \\ 3 \end{bmatrix}_2 = \frac{(2^4 - 1)(2^3 - 1)(2^2 - 1)}{(2^3 - 1)(2^2 - 1)(2 - 1)} = 15 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}_2.$$ 

That is, there are equally many points and blocks.

6.4 Geometries over $\text{GF}(2)$ and Sylvester Hadamard matrices

The case $F = \text{GF}(2)$ is especially interesting. In this case, a 1-dimensional subspace of a vector space $V$ over $F$ has just two vectors, the zero vector and a non-zero vector $v$. So there is a bijection between 1-dimensional subspaces of $V$ and the set $V \setminus \{0\}$.

If $\dim(V) = n$, there is also a bijection between $(n-1)$-dimensional subspaces of $V$ and the set $V \setminus \{0\}$. For any $(n-1)$-dimensional subspace is the kernel of a unique element of the dual space $V^*$ of $V$, the space of linear maps from $V$ to $F$. But any linear map can be represented in the form $f(v) = v.a$ for a fixed vector $a \in V$, so we can write $f = f_a$, and note that $f_1 \neq 0$ if and only if $a \neq 0$.

So we do the following construction, including the zero vector as well. Given $n$, let $H$ be the $2^n \times 2^n$ matrix whose rows and columns are indexed by all the vectors in $V$, the $n$-dimensional vector space over $F$. The $(v,w)$ entry of $H$ is defined to be $(-1)^{v \cdot w}$; in other words, it is +1 if $v \cdot w = 0$ and −1 if $v \cdot w = 1$. 

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Theorem 6.9 \( H \) is a Hadamard matrix of Sylvester type.

Proof Let \( v_1 \) and \( v_2 \) be two distinct vectors in \( V \), and consider the rows labelled by \( v_1 \) and \( v_2 \). Their entries in column \( w \) agree if and only if \( v_1.w = v_2.w \), that is, \( (v_1 - v_2).w = 0 \). So the two rows agree in the \( 2^{n-1} \) positions of the \((n-1)\)-dimensional subspace \( \{ w : (v_1 - v_2).w = 0 \} \), and disagree in the complementary set of \( 2^{n-1} \) positions. Since all entries in \( H \) are \( \pm 1 \), we see that any two rows are orthogonal, and so \( H \) is a Hadamard matrix.

We see that it is a Sylvester matrix by induction. Let \( W \) be the subspace of \( V \) consisting of vectors with last entry 0. Let \( H' \) be the matrix of order \( 2^{n-1} \) obtained from the above construction by using only the vectors in \( W \). We order the rows and columns of \( H \) by putting first the vectors \((v,0)\) in \( W \), and then the vectors \((v,1) \notin W \). We have

\[
(v,0).(w,0) = (v,0).(w,1) = (v,1).(w,0) = (v,1).(w,1) - 1,
\]

so we find that

\[
H = \begin{pmatrix} H' & H' \\ H' & -H' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes H'.
\]

By the inductive hypothesis, \( H' \) is a Sylvester matrix of order \( 2^{n-1} \); so \( H \) is a Sylvester matrix of order \( 2^n \).

Now how is this connected with the designs we constructed earlier?

We see that the first row and column of \( H \) consists entirely of entries \(+1\), since \( v_0 = 0, v = 0 \) for all \( v \in V \). Now, if we remove the first row and column and replace entries \(-1\) by 0, we obtain a matrix \( A \) whose rows and columns are indexed by the non-zero vectors in \( V \), with \( (v,w) \) entry 1 if \( v.w = 0 \) and 0 if \( v.w = 1 \). In other words, the rows are indexed by non-zero vectors, and the columns by non-zero linear maps from \( V \) to \( F \), so that the positions of the 1s in column \( f \) are the points in the corresponding hyperplane.

In other words, the matrix \( A \) is the incidence matrix of the \( 2-(2^n - 1,2^{n-1} - 1,2^{n-2} - 1) \) design of points and hyperplanes in \( PG(n - 1,2) \).

For \( n = 3 \), this design is the Fano plane. Thus we have “reverse engineered” the relation between the Fano plane and the Hadamard matrix of order 8 which we saw in lectures.

Exercise Recall that a Hadamard matrix is normalised if the first row and column consist entirely of \(+1\)s.
Prove that, if \( H \) is a normalised Hadamard matrix of order \( 4m \), and \( A \) is the matrix obtained by deleting the first row and column and replacing entries \(-1\) by \(0\), then \( A \) is the incidence matrix of a \(2-(4m-1, 2m-1, m-1)\) design.

I will use the construction given in this section to answer an exercise from an earlier problem sheet, which asked:

Show that the Sylvester and Paley matrices of order 32 are not isomorphic.

**Proposition 6.10** Let \( H \) be a Sylvester Hadamard matrix of order \( n = 2^d \). Given any three distinct rows \( R_1, R_2, R_3 \) of \( H \), there is a fourth row \( R_4 \) such that the products of the four elements of these rows in any column is the same.

**Proof** We note that this property is unaffected by row and column permutations and sign changes. (Column permutations and sign changes leave the four-fold products unaltered; row permutations change the numbering, and row sign changes might change the sign of all the products or leave them all unaltered.)

So we take the Sylvester matrix with rows and columns indexed by vectors in \( V = (\mathbb{Z}/(2))^d \), with entry in row \( v \) and column \( w \) equal to \((-1)^{v.w}\).

Choose any three distinct vectors \( v_1, v_2, v_3 \), and let \( v_4 = v_1 + v_2 + v_3 \). Observe that \( v_4 \) is different from all of \( v_1, v_2, v_3 \): if, say, \( v_4 = v_1 \), then \( v_2 = v_3 \), contrary to assumption. Now the product of the four elements in column \( w \) in these rows is

\[
(-1)^{v_1.w} \cdot (-1)^{v_2.w} \cdot (-1)^{v_3.w} \cdot (-1)^{v_4.w} = (-1)^{(v_1 + v_2 + v_3 + v_4).w} = +1.
\]

So to show that this is not equivalent to the Paley matrix \( P(31) \), it suffices to show that there are three rows of this matrix whose elementwise product is not another row or the negative of one. This can be done by writing out three rows of the matrix (say the first three) and forming their pairwise product, and observing that it is not a shift of the first row.

### 7 Recognising projective spaces

I will conclude with a little bit on this topic.

A very important result in the foundations of geometry is Desargues’ Theorem, see Figure 1.
Theorem 7.1 Given ten points, satisfying nine of the ten collinearities shown in Figure 1, the tenth collinearity is also satisfied.

I will not give a proof of this theorem, which can be done by calculations with coordinates in the field $F$. (Desargues’ original theorem dealt with the case $F = \mathbb{R}$.) However, it is not hard to see that it is true “generically” in three dimensions. We are trying to show that $p$, $q$, $r$ are collinear. Imagine that we have three poles tied together at the point $o$, and resting on the ground at the points $a_2$, $b_2$ and $c_2$. Now take a plane cutting the poles at the points $a_1$, $b_1$, $c_1$. This plane will intersect the ground plane in a line which contains all three points $p$, $q$ and $r$.

The more interesting and important theorem is the converse:

Theorem 7.2 A projective plane which satisfies Desargues’ Theorem is isomorphic to $\text{PG}(2,F)$ for some (possibly non-commutative) field $F$. In particular, a finite projective plane satisfying Desargues’ Theorem is $\text{PG}(2,q)$ for some prime power $q$.

The last part works because of Wedderburn’s Theorem, which says there are no non-commutative finite fields.
Theorem 7.3 (Wedderburn’s Theorem) A finite division ring is commutative (and so is a Galois field).

There is a difficulty here. The Fano plane has only seven points: how can it satisfy Desargues’ Theorem when the figure has ten points? The answer is that the figure can be allowed to degenerate; certain points are allowed to coincide with other points (but this has to be very carefully specified!).

For higher-dimensional projective spaces, the position is simpler.

The smallest example, after projective planes, is the Steiner system of points and lines in PG(3, 2), which is a $S(2, 3, 15)$. It has been known for nearly a century that there are 80 different (that is, non-isomorphic) Steiner systems $S(2, 3, 15)$. How do we recognise the projective geometry?

One property which does this is the fact that, in the projective geometry, three non-collinear points span a plane, which is isomorphic to the Fano plane (Figure 2).

![Figure 2: The Fano plane](image)

This has another consequence which is simpler to describe. A Steiner system is said to satisfy Pasch’s Axiom if, whenever two lines $L$ and $L'$ meets each of two intersecting lines $M$ and $M'$, not at their point of intersection, then $L$ and $L'$ intersect (Figure 3).

![Figure 3: Pasch’s Axiom](image)
Theorem 7.4 The following three conditions are equivalent for a Steiner system $S(2,3,n)$:

(a) $S$ is isomorphic to the point-line design in a projective geometry over $GF(2)$;
(b) any three non-collinear points in $S$ lie in a Fano plane;
(c) $S$ satisfies Pasch’s Axiom.

Proof We already remarked that (a) implies (b). To see that (b) implies (c), we note that the two intersecting lines $M$ and $M'$ lie in a Fano plane, and hence $L$ and $L'$ are also lines of this plane; but any two lines of the Fano plane intersect.

Suppose that (c) holds. Let $P$ be the set of points, and let $A = P \cup \{0\}$, where $0$ is a new symbol not in $P$. We define an operation of “addition” on $A$ by the rules

\[
p + 0 = 0 + p = p, \quad p + p = 0;
p + q = r \text{ if } p \neq q;
\]

where, in the second line, $\{p, q, r\}$ is the line containing $p$ and $q$.

We claim that $(A, +)$ is an abelian group. Clearly the operation is commutative; $0$ is the identity element; and every element is its own inverse. The difficulty is to prove the associative law

\[
(p + q) + s = p + (q + s)
\]

for all $p, q, s$. The cases where one of $p, q, s$ is zero, or where they all lie in a block, are just routine checking; so suppose not. Taking $p, q, s$ to be the points shown, we see that the associative law follows from Pasch’s Axiom (Figure 4).

![Figure 4: The associative law](image)

Now we define scalar multiplication on $A$, with scalar field $GF(2)$. This is easy: $0 \cdot a = 0$ and $1 \cdot a = a$ for all $a \in A$.

It is routine to show that $A$ is now a vector space over $GF(2)$. The points of the corresponding projective space are its 1-dimensional subspaces, which have
the form \( \{0, p\} \) for \( p \in P \); and the lines are its 2-dimensional subspaces, which have the form \( \{0, p, q, r\} \), where \( p + q = r \). This exactly matches up with the given Steiner triple system. So (a) holds.

This theorem is true in much greater generality. Let us temporarily define a linear space to be a structure with points and lines, where each line has at least three points, and any two points lie on a unique line. We make a technical assumption about “finite dimension” which is quite difficult to state\(^1\) and will not concern us since we are only interested in the finite case.

**Theorem 7.5** Let \( \mathcal{L} \) be a linear space. Then the following three conditions are equivalent:

(a) \( \mathcal{L} \) consists of the points and lines of \( \text{PG}(n, F) \), for some (possibly non-commutative) field \( F \), or \( \mathcal{L} \) is a projective plane;

(b) any three non-collinear points are contained in a projective plane;

(c) Pasch’s Axiom holds.

Let us just briefly note two things here:

- The theorem speaks of (possibly non-commutative) fields, sometimes known as division rings. One can do linear algebra over a division ring, and build a projective geometry from the subspaces of a vector space. Most properties work as in the commutative case (but duality may fail: the dual structure is a projective space, but over the “opposite” division ring, with \( a \circ b = ba \)).

- There are many projective planes which do not arise from vector spaces over fields, commutative or otherwise. Indeed, we saw already the following result:

**Theorem 7.6** A projective plane arises from a vector space over a (possibly non-commutative) field if and only if it satisfies Desargues’ Theorem.

We saw at least a plausibility argument for the fact that Desargues’ Theorem holds in projective spaces of dimension greater than 2. This is an important step in deriving the characterisation of projective geometries from the theorem above.

In the finite case, we have a characterisation of finite projective geometries:

\(^1\)Define a subspace of a linear space to be a set of points which contains the line through any two of its points. The required condition is that any chain of subspaces (ordered by inclusion) has length at most \( d \), for some natural number \( d \).
**Theorem 7.7** For a Steiner system $S = S(2, k, n)$, with $2 < k < n$, the following are equivalent:

(a) $S$ is the structure of points and lines in $\text{PG}(d, q)$ for some $d, q$, or $S$ is a projective plane;

(b) any three non-collinear points lie in a projective plane;

(c) Pasch’s Axiom holds.

### 7.1 Automorphism groups

This material will not be covered in lectures. It is included as supplementary material.

The so-called *Fundamental Theorem of Projective Geometry* tells us, among other things, about the group of automorphisms of a projective geometry. We now outline the situation in the finite case.

By an *automorphism* of $\text{PG}(n, q)$, we mean a permutation of the point set which maps lines to lines, planes to planes, and so on. As usual, the collection of all automorphisms forms a group, with the operation of composition.

You may wish to pause at this point and turn back to the Fano plane, and convince yourself that this geometry has 168 automorphisms.

The elements of the projective geometry $\text{PG}(n, q)$ are subspaces in an $(n + 1)$-dimensional vector space $V$ over $\text{GF}(q)$. It is clear that any invertible linear mapping of $V$ permutes the subspaces among themselves and preserves dimension, so induces an automorphism. The group of invertible linear mappings of $V$ is the *general linear group*, denoted by $\text{GL}(V)$ or $\text{GL}(n + 1, q)$.

However, different linear maps may induce the same automorphism. We have to compute the *kernel* of the action of $\text{GL}(n, F)$ on the set of points of the projective space, the subgroup consisting of maps which fix all the points.

**Theorem 7.8** The kernel of the action of $\text{GL}(n, q)$ on the points of the projective space is the subgroup of scalar matrices $Z = \{cI : c \in \text{GF}(q), c \neq 0\}$.

**Proof** Suppose that $A$ is a linear map which fixes every 1-dimensional subspace of $V$. Take a basis $(e_1, e_2, \ldots, e_{n+1})$ for $V$. Since $\langle e_1 \rangle$ is fixed, we have $e_1A = c_1e_1$ for some scalar $c_1$. Similarly, for each $i$, $e_iA = c_ie_i$. Now let $f = e + 1 + \cdots + e_{n+1}$. Then $fA = df$ for some scalar $d$. But then

$$c_1e_1 + \cdots + c_{n+1}e_{n+1} = (e_1 + \cdots + e_{n+1})A = d(e_1 + \cdots + e_{n+1}),$$

and so by linear independence we have $c_1 = \cdots = c_{n+1} = d$.
Thus the group of automorphisms of projective space induced by \( \text{GL}(n+1, q) \) is the quotient group \( \text{GL}(n+1, q)/\mathbb{Z} \), which we call the \textit{projective general linear group} \( \text{PGL}(n+1, q) \).

Further automorphisms arise from field automorphisms acting coordinatewise on the vector space. It is not hard to see that these maps do indeed preserve subspaces and dimension, although they are not linear.) Recall that, if \( q = p^s \) (\( p \) prime), then the group of automorphisms of \( \text{GF}(q) \) is cyclic of order \( s \), generated by the \textit{Frobenius automorphism} \( a \mapsto a^p \).

\textbf{Theorem 7.9} \textit{The automorphism group of} \( \text{PG}(n, q) \) \textit{for} \( n \geq 2 \) \textit{is the group generated by} \( \text{PGL}(n+1, q) \) \textit{and the field automorphisms of} \( \text{GF}(q) \).

We will compute the order of this group. A \( (n+1) \times (n+1) \) matrix is invertible if and only if its rows are linearly independent. Our earlier counting argument shows that the number of such matrices is

\[ |\text{GL}(n+1, q)| = (q^{n+1} - 1)(q^{n+1} - q)(q^{n+1} - q^2)\cdots(q^{n+1} - q^n). \]

We have to divide by \( |\mathbb{Z}| \), the number of non-zero scalars, to get

\[ |\text{PGL}(n+1, q)| = \frac{|\text{GL}(n+1, q)|}{q-1}. \]

Finally we have to multiply by the order \( s \) of the group of field automorphisms (where \( q = p^s \)) to get the order of the group of automorphisms of \( \text{PG}(n, q) \). This group is denoted by \( \text{PGL}(n+1, q) \). (The Greek letter indicates that field automorphisms are included.)

As a check, the order of \( \text{GL}(3, 2) \) is \((2^3 - 1)(2^3 - 2)(2^3 - 2^2) = 168 \). The group of nonzero scalars and the group of field automorphisms both have order 1. So the automorphism group of the Fano plane does indeed have order 168.