1. In this question, a Steiner system $S(2, 3, 7)$ refers to a set of 7 points, with a collection of 3-element subsets called blocks, such that any two distinct points lie in a unique block. You may assume without proof that such a Steiner system is unique up to isomorphism.

(a) State the sphere-packing bound for an error-correcting code.

(b) Let $C$ be a linear binary $[7, 4, 3]$ code. Show that any word in $\{0, 1\}^7$ lies at distance at most 1 from a unique word in $C$.

(c) Deduce that the positions supporting words of weight 3 are the blocks of a Steiner system $S(2, 3, 7)$.

(d) Hence prove the uniqueness of $C$ up to coordinate permutations, and write down its weight enumerator.

(e) Describe briefly a syndrome decoding scheme for the code $C$.

2. (a) State the MacWilliams theorem connecting the weight enumerators of a linear code and its dual.

(b) Let $C$ be a binary linear code with weight enumerator

$$X^7 + 7X^4Y^3 + 7X^3Y^4 + Y^7.$$ 

Show that $C$ is a $[7, 4, 3]$ code and deduce that it is 1-error correcting.

(c) Prove that the words of even weight in $C$ form a subcode which is equal to $C^\perp$.

(d) Verify the conclusions of MacWilliams’ Theorem for this pair of codes.

(e) Let $A$ be the $3 \times 7$ matrix whose columns are the base 2 representations of the integers 1, 2, \ldots, 7. Show that the code with parity check matrix $A$ has the weight enumerator given in part (a).
3. Let $X$ be the graph shown below.

\[\text{\includegraphics{graph.png}}\]

(a) Write down the chromatic polynomial of $X$.

(b) What is the chromatic number of $X$?

(c) Describe the automorphism group $G$ of $X$. What is its order?

(d) Calculate the orbital chromatic polynomial of $X$ and $G$.

(e) What is meant by an \textit{acyclic orientation}? How many acyclic orientations does $X$ have?

4.

(a) In how many ways can the vertices of a regular hexagon be coloured with $q$ colours?

(b) How many colourings are there if we regard colourings which differ by a rotation of the hexagon as the same?

(c) How many of the colourings in (a) satisfy the requirement that adjacent vertices must have different colours?

(d) How many colourings satisfying this requirement are there if we regard colourings which differ by a rotation of the hexagon as the same?
5.

(a) What is a Gray code?

(b) Show that a Gray code of length 4 gives an isometry between $\mathbb{Z}_4$ (with the Lee metric), and $\mathbb{Z}_2^2$ (with the Hamming metric), defining these terms. Is it a linear map?

(c) Show how to extend this to an isometry from $\mathbb{Z}_4^n$ (with the Lee metric) to $\mathbb{Z}_2^{2n}$ (with the Hamming metric).

(d) Let $C$ be the $\mathbb{Z}_4$ linear code of length 4 spanned by $(1,1,1,1)$ and $(0,1,2,3)$. Show that the Gray map image of $C$ is a binary code of length 8 and minimum distance 4, containing 16 codewords.

(e) Is the binary code just defined linear? (Proof not required.)

6.

(a) Define a matroid, in terms of its independent sets.

(b) What are the independent sets of the uniform matroid $U_{4,2}$?

(c) Show that $U_{4,2}$ is not a graphic matroid, explaining this term.

(d) Write down vectors representing the matroid $U_{4,2}$ over the field $\mathbb{Z}_3$.

(e) Show that the corresponding code over $\mathbb{Z}_3$ attains equality in either the sphere-packing bound or the Singleton bound. (You should state the bound but need not prove it.)
7.

(a) Define a matroid, in terms of its independent sets.

(b) What are the independent sets of the graphic matroid $M(K_4)$, where $K_4$ is the complete graph on four vertices.

(c) Is this matrix representable over a finite field? If so, find a representation; if not, explain why.

8.

(a) Define the Tutte polynomial of a matroid.

(b) Explain in detail how

    (i) the chromatic polynomial of a graph, and
    (ii) the weight enumerator of a linear code

    are specialisations of the Tutte polynomial of suitable matroids. Proof is not required.

(c) Find the Tutte polynomial of a free matroid (one all of whose subsets are independent), and hence find the chromatic polynomial of a tree on $n$ vertices.

(d) Define the dual of a matroid, and state the relation between the Tutte polynomials of a matroid and its dual. (Proof not required.)
9.

(a) State the Orbit-counting Lemma.

(b) What is the cycle index of a permutation group? State the cycle index theorem.

(c) Compute the cycle index of the group of rotations and reflections of a regular hexagon (acting on the vertices of the hexagon).

(d) Hence compute a polynomial $f(x)$ of degree 6, in which the coefficient of $x^i$ is equal to the number of colourings of the vertices with colours red and blue having $i$ red vertices, up to symmetries of the hexagon.

10.

(a) State and prove the Orbit-counting Lemma for a permutation group.

(b) What is meant by saying that a permutation group is transitive? Show that a transitive finite permutation group acting on a set with more than one point contains an element which has no fixed points.

(c) Deduce that if $H$ is a proper subgroup of a finite group $G$, then there is an element of $G$, none of whose conjugates lies in $H$. (Hint: Let $G$ act on the set of right cosets of $H$ by right multiplication.)

(d) Give an example of a permutation group in which every element has a fixed point.