Generalisations of EPPO groups using graphs

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The organisers were Vijayakumar Ambat and Aparna Lakshmanan at CUSAT, to whom I am grateful.
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- Of the two groups of order 6, the dihedral group is an EPPO group (all elements have orders 1, 2 or 3) but the cyclic group is not.
- Thinking about this example, we see that a nilpotent group (which is the direct product of its Sylow subgroups) is an EPPO group if and only if it has prime power order.
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Theorem
An EPPO group $G$ satisfies one of the following:

1. $|\pi(G)| = 1$ and $G$ is a $p$-group.
2. $|\pi(G)| = 2$ and $G$ is a soluble Frobenius or $2$-Frobenius group (see later).
3. $|\pi(G)| = 3$ and $G \in \{A_6, \text{PSL}_2(7), \text{PSL}_2(17), M_{10}\}$.
4. $|\pi(G)| = 3$, $G/O_2(G)$ is $\text{PSL}_2(2^n)$ for $n \in \{2, 3\}$ and if $O_2(G) \neq \{1\}$, then $O_2(G)$ is the direct product of minimal normal subgroups of $G$, each of which is of order $2^{2n}$ and as $G/O_2(G)$-module is isomorphic to the natural $GF(2^n)\text{SL}_2(2^n)$-module.
5. $|\pi(G)| = 4$ and $G \cong \text{PSL}_3(4)$.
6. $|\pi(G)| = 4$, $G/O_2(G)$ is $\text{Sz}(2^n)$ for $n \in \{3, 5\}$, and if $O_2(G) \neq \{1\}$, then $O_2(G)$ is the direct product of minimal normal subgroups of $G$, each of which is of order $2^{4n}$ and as $G/O_2(G)$-module is isomorphic to the natural $GF(2^n)\text{Sz}(2^n)$-module of dimension $4$. 

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- A glance at the ATLAS of finite groups shows, for example, that the Mathieu group $M_{11}$ has vertex set \{2, 3, 5, 11\} and just a single edge \{2, 3\}.
- $G$ is an EPPO group if and only if its GK graph is a null graph (that is, has no edges).
Frobenius and 2-Frobenius groups

The group $G$ is a **Frobenius group** if it has a proper subgroup $H$ (called a **Frobenius complement**) with the property that $H \cap H^g = \{1\}$ for all $g \in G \setminus H$. The symmetric group $S_3$ is an example.
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The symmetric group $S_4$ is an example.
The main theorem of Gruenberg and Kegel was a structure theorem for groups whose GK graph is disconnected. It was contained in an unpublished manuscript, and published by J. S. Williams (a student of Gruenberg) in 1981.

**Theorem**

Let $G$ be a finite group whose GK graph is disconnected. Then one of the following holds:

1. $G$ is a Frobenius or $2$-Frobenius group;
2. $G$ is an extension of a nilpotent $\pi$-group by a simple group by a $\pi$-group, where $\pi$ is the set of primes in the connected component containing $2$.

Which simple groups can occur in the second conclusion of the theorem? This was investigated by Williams, though he was unable to deal with groups of Lie type in characteristic 2. The work was completed by Kondrat'ev in 1989, and some errors corrected by Kondrat'ev and Mazurov in 2000.
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GK graph and EPPO groups

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Other graphs

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- $g$ and $h$ are joined in the **power graph** $\text{Pow}(G)$ of $G$ if one of them is a power of the other.

- $g$ and $h$ are joined in the **enhanced power graph** $\text{EPow}(G)$ of $G$ if they are both powers of an element $k$ (in other words, if the group $\langle g, h \rangle$ they generate is cyclic).
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We see that the edge set of $\text{Pow}(G)$ is contained in that of $\text{EPow}(G)$. (In graph theory language, $\text{Pow}(G)$ is a spanning subgraph of $\text{EPow}(G)$.)
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The finite group $G$ satisfies $\text{Pow}(G) = \text{EPow}(G)$ if and only if $G$ is an EPPO group.
EPPO groups reappear

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**Proof.**

If $G$ fails to be an EPPO group, then it contains an element $g$ of order $pq$ for some primes $p$ and $q$. Then $g^p$ and $g^q$ are joined in the enhanced power graph (since both are powers of $g$) but not in the power graph.

If $G$ is an EPPO group, and $\langle g, h \rangle$ is cyclic, then it has prime power order, and so one of $g$ and $h$ generates this group, say $g$. Then $h$ is a power of $g$. Thus the classification of EPPO groups gives us the groups $G$ for which $\text{Pow}(G) = \text{EPow}(G)$. 
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\hfill $\Box$
Generalisations

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Let \( p \) be any graph parameter which is monotonic: that is, adding edges to a graph cannot decrease the value of \( p \). Then \( p(\text{Pow}(G)) \leq p(\text{EPow}(G)) \). Asking for which groups equality holds is a generalisation of asking for which groups \( \text{Pow}(G) = \text{EPow}(G) \), that is, the EPPO groups.

Here is a fairly easy example. The clique number of a graph is the size of the largest complete subgraph.

**Theorem**

For a finite group \( G \), the power graph and enhanced power graph have equal clique number if and only if the largest order of an element of \( G \) is a prime power. Clearly this class of groups includes the EPPO groups!
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- $q$ is a power of 2 and $q + 1$ is a Fermat prime;
- $q$ is a Mersenne prime and $q + 1$ is a power of 2;
- $q = 8$, $q + 1 = 9$.

(The Catalan conjecture asserts that the only solution of $x^a - y^b = 1$ in positive integers $x, y, a, b$ with $a, b > 1$ is $3^2 - 2^3 = 1$. It was proved by Mihăilescu in 2002.)
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(The Catalan conjecture asserts that the only solution of $x^a - y^b = 1$ in positive integers $x$, $y$, $a$, $b$ with $a$, $b > 1$ is $3^2 - 2^3 = 1$. It was proved by Mihăilescu in 2002.)
An example

For which prime powers $q$ do the power graph and enhanced power graph of $\text{PGL}(2, q)$ have the same clique number? The maximal order of an element in this group is $q + 1$, so the necessary and sufficient condition is that $q + 1$ is also a prime power. According to the Catalan conjecture, this occurs only in one of the following cases:

- $q$ is a power of 2 and $q + 1$ is a Fermat prime;
- $q$ is a Mersenne prime and $q + 1$ is a power of 2;
- $q = 8, q + 1 = 9$.

(The Catalan conjecture asserts that the only solution of $x^a - y^b = 1$ in positive integers $x, y, a, b$ with $a, b > 1$ is $3^2 - 2^3 = 1$. It was proved by Mihăilescu in 2002.)
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Matching number

I will illustrate with one somewhat striking example. The matching number $\mu(\Gamma)$ of a graph $\Gamma$ is the maximum number of pairwise disjoint edges in $\Gamma$. This is clearly a monotonic graph parameter.
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*For any finite group $G$, the matching numbers of $\text{Pow}(G)$ and $\text{EPow}(G)$ are equal.*
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The strategy of the proof is to show that, given a matching in the enhanced power graph, we can replace its edges by edges of the power graph to find another matching of the same size.
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Since the path $P_4$ is isomorphic to its complement, the class of cographs is self-complementary. In fact, it is the smallest class of graphs containing the 1-vertex graph and closed under disjoint union and complementation. This means that the class has very nice algorithmic properties, which don’t concern us here.
The power graph of a $p$-group is a cograph

Recall that in the power graph, $g$ and $h$ are joined if one is a power of the other. So the graph is naturally a directed graph, with an arc $g \to h$ if $h$ is a power of $g$. It is easily seen that this relation is transitive.
Recall that in the power graph, $g$ and $h$ are joined if one is a power of the other. So the graph is naturally a directed graph, with an arc $g \rightarrow h$ if $h$ is a power of $g$. It is easily seen that this relation is transitive. Hence, if we have an induced $P_4$ in a cograph, directions must alternate:

$$a \rightarrow b \leftarrow c \rightarrow d.$$
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Hence, if we have an induced \( P_4 \) in a cograph, directions must alternate:

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Now in a \( p \)-group, if \( c \rightarrow b \) and \( c \rightarrow d \), then \( b \) and \( d \) lie in a cyclic group of prime power order, so one is a power of the other. Hence there can be no induced \( P_4 \):

**Theorem**

The power graph of a group of prime power order is a cograph.
The power graph of an EPPO group is a cograph

This follows easily from the previous result. Hence the following problem is a generalisation of the problem of determining EPPO groups:

Problem
Determine the finite groups whose power graph is a cograph.

I have worked on this problem with Pallabi Manna and Ranjit Mehatari from Rourkela. Our first theorem states:

Theorem
If $G$ is a nilpotent group, then the power graph of $G$ is a cograph if and only if either $G$ has prime power order, or $G = C_{pq}$ where $p$ and $q$ are primes.
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Recall that a nilpotent EPPO group has prime power order. The addition of the groups $C_{pq}$ has a big effect on the class of groups!
Simple groups whose power graph is a cograph

Using this result, it is possible to show the following.

**Theorem**

*Let $G$ be a finite simple group whose power graph is a cograph. Then one of the following holds:*

- $G = \text{PSL}(2, q)$ for a prime power $q$, where each of $(q+1)\frac{\gcd(q+1,2)}{\gcd(q+1,2)}$ and $(q-1)\frac{\gcd(q-1,2)}{\gcd(q-1,2)}$ is either a prime power or the product of two primes;

- $G = \text{Sz}(q)$ for $q$ an odd power of 2, where each of $q-1$, $q+\sqrt{2}q+1$ and $q-\sqrt{2}q+1$ is either a prime power or the product of two primes;

- $G = \text{PSL}(3,4)$.

Note that $\text{PSL}(2,11)$ and $M_{11}$ have identical GK graphs, but the power graph of the first is a cograph, that of the second is not.
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Are there infinitely many values of $q$ for which $\text{Pow}(\text{PSL}(2, q))$ is a cograph?
A problem for number theorists

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For example, the values of $d$ up to 200 for which the power graph of $\text{PSL}(2, 2^d)$ is a cograph are 1, 2, 3, 4, 5, 7, 11, 13, 17, 19, 23, 31, 61, 101, 127, 167, 199.
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Similar (possibly easier) question for $\text{Sz}(q)$. 
More graphs

I will finish with a couple of recent results, not specifically about EPPO groups, but about the question: given two graphs on a group, for which groups do they coincide? We will see that other interesting classes of groups arise.
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Clearly the commuting graph contains the enhanced power graph as a subgraph. When are they equal? We saw that the enhanced power graph is equal to the power graph if and only if $G$ contains no $C_p \times C_q$ where $p$ and $q$ are distinct primes. Similarly, the commuting graph equals the enhanced power graph if and only if $G$ contains no $C_p \times C_p$, where $p$ is prime.
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From this, the groups can be determined.
Let $G$ be a group containing no $C_p \times C_p$. By a theorem of Burnside, all Sylow subgroups of $G$ are cyclic or generalized quaternion.
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- by the Brauer–Suzuki theorem, $H = G/O(G)$ has a unique subgroup $Z$ of order 2, where $O(G)$ is the largest normal subgroup of odd order;
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- a cohomological argument due to Glauberman shows that any such group $H/Z$ has a unique cover $H$ with unique subgroup of order 2;
- using the fact that the other Sylow subgroups are cyclic, it is possible to determine $G$. 
Following work by several authors, G. Arunkumar, Rajat Kanti Nath, Lavanya Selvaganesh and I defined, for each type of graph on a group $G$, a conjugacy supergraph, in which $g$ and $h$ are joined if and only if there are conjugates of $g$ and $h$ which are joined in the original graph.
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Super graphs

Following work by several authors, G. Arunkumar, Rajat Kanti Nath, Lavanya Selvaganesh and I defined, for each type of graph on a group $G$, a conjugacy supergraph, in which $g$ and $h$ are joined if and only if there are conjugates of $g$ and $h$ which are joined in the original graph. The idea can be extended to other equivalence relations on the group, such as “same orbit of the automorphism group”, or “same order”. Here are two theorems from our paper in preparation.
Two theorems

Theorem

For a finite group $G$, the following are equivalent:

$\begin{align*}
\text{▶} & \quad \text{the conjugacy supercommuting graph is equal to the commuting graph;} \\
\text{▶} & \quad \text{every centralizer is a normal subgroup;} \\
\text{▶} & \quad G \text{ is a } 2\text{-Engel group, that is, satisfies the identity } [y, x, x] = 1.
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The **generating graph** of a group $G$ has an edge from $g$ to $h$ whenever $\langle g, h \rangle = G$. It has been the subject of much research. Unfortunately, if $G$ is not 2-generated, then the generating graph is null. To overcome this defect, Lucchini defined the **independence graph** of $G$, in which $g$ and $h$ are joined if and only if $\{g, h\}$ is contained in a minimal (with respect to inclusion) generating set for $G$. Note that, if $h$ is a power of $g$, then $\{g, h\}$ is not contained in any minimal generating set. In a paper in preparation, Lucchini and Nemmi say that $G$ has the independence property if the converse holds, that is, the independence graph is the complement of the power graph.
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Lucchini and Nemmi determined the soluble groups with the independence property. They also showed that there are no non-soluble groups, using the following very recent theorem of Saul Freedman:

**Theorem**

Let $S$ be a non-abelian finite simple group. Then there exist non-commuting elements $s, x \in S$ such that, whenever $G$ is an almost simple group with socle $S$, and $M \in \mathcal{G}(s)$ denotes the set of maximal subgroups of $G$ containing $s$, then $x \in \bigcap_{M \in \mathcal{G}(s)} M$. 
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$$x \in \bigcap_{M \in \mathcal{M}_G(s)} M.$$
Another variation is to define the **rank graph** to have an edge \( \{g, h\} \) whenever \( \{g, h\} \) is contained in a generating set of minimal cardinality for \( G \), this minimal cardinality being the **rank** of \( G \). If the rank is 2, this is just the generating graph.
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References


- Peter J. Cameron, Pallabi Manna and Ranjit Mehatari, On finite groups whose power graph is a cograph, *J. Algebra* 591 (2022), 59–74.

- Peter J. Cameron, V. V. Swathi and M. S. Sunitha, Matching in power graphs of finite groups, arXiv 2107.01157

- Saul D. Freeman, Elements in non-conjugate subgroups of finite almost simple groups, in preparation.


- Andrea Lucchini and Daniele Nemmi, The power graph and the independence graph, in preparation.
... for your attention.