Generalisations of EPPO groups using graphs

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Ural Seminar on Group Theory and Combinatorics
Ural Federal University, Yekaterinburg
23 November 2021
EPPO groups

An **EPPO group** is a finite group in which every element has prime power order.

- A group of prime power order is an EPPO group.
- Of the two groups of order 6, the dihedral group is an EPPO group (all elements have orders 1, 2 or 3) but the cyclic group is not.
- Thinking about this example, we see that a nilpotent group (which is the direct product of its Sylow subgroups) is an EPPO group if and only if it has prime power order.

**History**

EPPO groups were introduced by Graham Higman in the 1950s; he classified the soluble ones.

In the 1960s, as a spin-off from the discovery of his infinite family of simple groups, Michio Suzuki classified the simple EPPO groups.

Earlier this year, I asked Natalia Maslova if she knew a classification of all EPPO groups. She sat down and produced one. I will tell you later why I wanted this.

**The Gruenberg–Kegel graph**

In the 1960s, while investigating integral representations of finite groups, Karl Gruenberg and Otto Kegel defined the **prime graph** of a finite group, now more usually referred to as the **Gruenberg–Kegel graph** or **GK graph**.

The vertex set of the GK graph of a group $G$ is the set of prime divisors of $|G|$. (Equivalently, by Cauchy’s Theorem, the set of prime orders of elements of $G$.) Two vertices $p$ and $q$ are joined if $G$ contains an element of order $pq$. This tiny graph carries a lot of information about the group.

- A glance at the ATLAS of finite groups shows, for example, that the Mathieu group $M_{11}$ has vertex set {2,3,5,11} and just a single edge (2,3).
- $G$ is an EPPO group if and only if its GK graph is a null graph (that is, has no edges).

**Frobenius and 2-Frobenius groups**

The group $G$ is a **Frobenius group** if it has a proper subgroup $H$ (called a **Frobenius complement**) with the property that $H \cap H^g = \{1\}$ for all $g \in G \setminus H$. The symmetric group $S_3$ is an example.

Frobenius showed that, if $N$ is the set of elements lying in no conjugate of $H$, together with the identity, then $N$ is a normal subgroup of $G$, called the **Frobenius kernel**. Moreover, Thompson showed that the Frobenius kernel is nilpotent, and Zassenhaus determined the structures of Frobenius complements.

The group $G$ is a **2-Frobenius group** if it has a chain of normal subgroups $\{1\} < N < M < G$ such that

- $M$ is a Frobenius group with Frobenius kernel $N$;
- $G/N$ is a Frobenius group with Frobenius kernel $M/N$.

The symmetric group $S_4$ is an example.

This project is one of several spin-offs from a research discussion on Graphs and Groups run from Kochi, India, over the summer.

The organisers were Vijayakumar Ambat and Aparna Lakshmanan at CUSAT, to whom I am grateful.
The main theorem of Gruenberg and Kegel was a structure
theorem for groups whose GK graph is disconnected. It was
contained in an unpublished manuscript, and published by
J. S. Williams (a student of Gruenberg) in 1981.

Theorem
Let G be a finite group whose GK graph is disconnected. Then one of
the following holds:

- G is a Frobenius or 2-Frobenius group;
- G is an extension of a nilpotent \( \pi \)-group by a simple group by a
\( \pi \)-group, where \( \pi \) is the set of primes in the connected
component containing 2.

Which simple groups can occur in the second conclusion of the
theorem? This was investigated by Williams, though he was
unable to deal with groups of Lie type in characteristic 2. The
work was completed by Kondrat’ev in 1989, and some errors
corrected by Kondrat’ev and Mazurov in 2000.

We have seen that G is an EPPO group if and only if its
Gruenberg–Kegel graph has no edges.

The finite group G satisfies Pow\((G) = \text{EPow}(G)\) if and only if G is
an EPPO group.

Proof.
If G fails to be an EPPO group, then it contains an element \( g \) of
order \( pq \) for some primes \( p \) and \( q \). Then \( g^p \) and \( g^q \) are joined in
the enhanced power graph (since both are powers of \( g \)) but not
in the power graph.

Conversely, if G is an EPPO group, and \((g,h)\) is cyclic, then it
has prime power order, and so one of \( g \) and \( h \) generates this
group, say \( g \); then \( h \) is a power of \( g \).

Thus the classification of EPPO groups gives us the groups G
for which Pow\((G) = \text{EPow}(G)\).

For which prime powers \( q \) do the power graph and enhanced
power graph of PGL(2, \( q \)) have the same clique number?

An example
For which prime powers \( q \) do the power graph and enhanced
power graph of PGL(2, \( q \)) have the same clique number?

The minimal order of an element in this group is \( q + 1 \), so the
necessary and sufficient condition is that \( q + 1 \) is also a prime
power.

According to the Catalan conjecture, this occurs only in one of
the following cases:

- \( q \) is a power of 2 and \( q + 1 \) is a Fermat prime;
- \( q \) is a Mersenne prime and \( q + 1 \) is a power of 2;
- \( q = 8, q + 1 = 9 \).

(The Catalan conjecture asserts that the only solution of
\( x^n - y^n = 1 \) in positive integers \( x, y, a, b \) with \( a, b > 1 \) is
\( 3^2 - 2^3 = 1 \). It was proved by Mihailescu in 2002.)
### Matching number

I will illustrate with one somewhat striking example. The **matching number** \( \mu(\Gamma) \) of a graph \( \Gamma \) is the maximum number of pairwise disjoint edges in \( \Gamma \). This is clearly a monotonic graph parameter.

With V. V. Swathi and M. S. Sunitha from Calicut, I proved:

**Theorem**

*For any finite group \( G \), the matching numbers of \( \text{Pow}(G) \) and \( \text{EPow}(G) \) are equal.*

The small surprise is that we cannot calculate the matching number of \( \text{Pow}(G) \) for all groups \( G \), merely give upper and lower bounds.

The strategy of the proof is to show that, given a matching in the enhanced power graph, we can replace its edges by edges of the power graph to find another matching of the same size.

### Cographs

To describe the other generalisation, we have to make a detour.

A **cograph** is a graph which doesn’t contain the 4-vertex path as an induced subgraph. These are also referred to as complement-reducible graphs, hereditary Dacey graphs, or (my favourite) N-free graphs. The variety of names indicate the importance of this class.

Since the path \( P_4 \) is isomorphic to its complement, the class of cographs is self-complementary. In fact, it is the smallest class of graphs containing the 1-vertex graph and closed under disjoint union and complementation. This means that the class has very nice algorithmic properties, which don’t concern us here.

### The power graph of a \( p \)-group is a cograph

Recall that in the power graph, \( g \) and \( h \) are joined if one is a power of the other. So the graph is naturally a directed graph, with an arc \( g \to h \) if \( h \) is a power of \( g \). It is easily seen that this relation is transitive.

Hence, if we have an induced \( P_4 \) in a cograph, directions must alternate:

\[ a \to b \leftarrow c \to d. \]

Now in a \( p \)-group, if \( c \to b \) and \( c \to d \), then \( b \) and \( d \) lie in a cyclic group of prime power order, so one is a power of the other. Hence there can be no induced \( P_4 \):

**Theorem**

The power graph of a group of prime power order is a cograph.

### The power graph of an EPPO group is a cograph

This follows easily from the previous result. Hence the following problem is a generalisation of the problem of determining EPPO groups:

**Problem**

Determine the finite groups whose power graph is a cograph.

I have worked on this problem with Pallabi Manna and Ranjit Mehatari from Rourkela. Our first theorem states:

**Theorem**

If \( G \) is a nilpotent group, then the power graph of \( G \) is a cograph if and only if either \( G \) has prime power order, or \( G = C_{pq} \) where \( p \) and \( q \) are primes.

Recall that a nilpotent EPPO group has prime power order. The addition of the groups \( C_{pq} \) has a big effect on the class of groups!

### Simple groups whose power graph is a cograph

Using this result, it is possible to show the following:

**Theorem**

Let \( G \) be a finite simple group whose power graph is a cograph. Then one of the following holds:

- \( G = \text{PSL}(2, q) \) for a prime power \( q \), where each of \( (q + 1)/\gcd(q + 1, 2) \) and \( (q - 1)/\gcd(q - 1, 2) \) is either a prime power or the product of two primes;
- \( G = \text{Sz}(q) \) for \( q \) an odd power of 2, where each of \( q - 1 \), \( q + \sqrt{q + 1} \) and \( q - \sqrt{q + 1} \) is either a prime power or the product of two primes;
- \( G = \text{PSL}(3, 4) \).

Note that \( \text{PSL}(2, 11) \) and \( M_{11} \) have identical GK graphs, but the power graph of the first is a cograph, that of the second is not.

### A problem for number theorists

**Problem**

Are there infinitely many values of \( q \) for which \( \text{Pow}(\text{PSL}(2, q)) \) is a cograph?

For example, the values of \( d \) up to 200 for which the power graph of \( \text{PSL}(2, 2^d) \) is a cograph are 1, 2, 3, 4, 5, 7, 11, 13, 17, 19, 23, 31, 61, 101, 127, 167, 199.

Similar (possibly easier) question for \( \text{Sz}(q) \).
More graphs

I will finish with a couple of recent results, not specifically about EPPO groups, but about the question: given two graphs on a group, for which groups do they coincide? We will see that other interesting classes of groups arise.

The commuting graph of a group $G$ has vertex set $G$, with an edge from $x$ to $y$ whenever $xy = yx$. This is the oldest of these graphs, appearing implicitly in the seminal 1955 paper of Brauer and Fowler.

Clearly the commuting graph contains the enhanced power graph as a subgraph. When are they equal?

We saw that the enhanced power graph is equal to the power graph if and only if $G$ contains no $C_p \times C_q$ where $p$ and $q$ are distinct primes. Similarly, the commuting graph equals the enhanced power graph if and only if $G$ contains no $C_p \times C_q$, where $p$ is prime.

From this, the groups can be determined.

Let $G$ be a group containing no $C_p \times C_p$. By a theorem of Burnside, all Sylow subgroups of $G$ are cyclic or generalized quaternion.

If they are all cyclic, then $G$ is metacyclic and its structure is clear.

If the Sylow 2-subgroups are generalized quaternion, then

- by the Brauer–Suzuki theorem, $H = G/O(G)$ has a unique subgroup $Z$ of order 2, where $O(G)$ is the largest normal subgroup of odd order;
- $H/Z$ has dihedral Sylow 2-subgroups, so is determined by the Gorenstein–Walter theorem;
- a cohomological argument due to Glauberman shows that any such group $H/Z$ has a unique cover $H$ with unique subgroup of order 2;
- using the fact that the other Sylow subgroups are cyclic, it is possible to determine $G$.

Super graphs

Following work by several authors, G. Arunkumar, Rajat Kanti Nath, Lavanya Selvaganesh and I defined, for each type of graph on a group $G$, a conjugacy supergraph, in which $g$ and $h$ are joined if and only if there are conjugates of $g$ and $h$ which are joined in the original graph.

The idea can be extended to other equivalence relations on the group, such as “same orbit of the automorphism group”, or “same order”.

Here are two theorems from our paper in preparation.

Two theorems

Theorem

For a finite group $G$, the following are equivalent:

- the conjugacy supercommuting graph is equal to the commuting graph;
- every centralizer is a normal subgroup;
- $G$ is a 2-Engel group, that is, satisfies the identity $[y, x, x] = 1$.

Theorem

For a finite group $G$, the following are equivalent:

- the conjugacy superpower graph is equal to the power graph;
- the conjugacy superenhanced power graph is equal to the enhanced power graph;
- $G$ is a Dedekind group, that is, every subgroup is normal.

The independence graph

I would like to finish with some connections between the power graph and enhanced power graph and some other graphs introduced by Andrea Lucchini, related to the generating graph.

The generating graph of a group $G$ has an edge from $g$ to $h$ whenever $(g, h) = G$. It has been the subject of much research. Unfortunately, if $G$ is not 2-generated, then the generating graph is null. To overcome this defect, Lucchini defined the independence graph of $G$, in which $g$ and $h$ are joined if and only if $(g, h)$ is contained in a minimal (with respect to inclusion) generating set for $G$.

Note that, if $h$ is a power of $g$, then $(g, h)$ is not contained in any minimal generating set. In a paper in preparation, Lucchini and Nemmi say that $G$ has the independence property if the converse holds, that is, the independence graph is the complement of the power graph.

Lucchini and Nemmi determined the soluble groups with the independence property. They also showed that there are no non-soluble groups, using the following very recent theorem of Saul Freedman:

Theorem

Let $S$ be a non-abelian finite simple group. Then there exist non-commuting elements $s, x \in S$ such that, whenever $G$ is an almost simple group with socle $S$, and $M_C(s)$ denotes the set of maximal subgroups of $G$ containing $s$, then $x \in \bigcap_{M \in M_C(s)} M$.  


Another variation is to define the rank graph to have an edge \( \{g, h\} \) whenever \( \{g, h\} \) is contained in a generating set of minimal cardinality for \( G \), this minimal cardinality being the rank of \( G \). If the rank is 2, this is just the generating graph. Just as edges of the power graph cannot be joined in the independence graph, so edges of the enhanced power graph cannot be joined in the rank graph. So, as above, we call \( G \) rank perfect if the rank graph is the complement of the enhanced power graph. Lucchini has shown that a rank perfect group must be supersoluble, and classified the non-nilpotent groups with this property.

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