Running a research discussion in South India

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You may have wondered what your teachers get up to during the vacation: is it like the novels of David Lodge, jetting all over the world for conferences and having amorous adventures? During the last couple of years, how did we manage without being able to do all that? The pandemic has a devastating effect on many people, who have lost loved ones or livelihoods, been afflicted with long Covid, or lost business or life opportunities.

I would like to tell you the story of a small good effect of Covid. It drove lectures, seminars and conferences on-line, but in this way made possible long-distance interactions and collaborations which would just not have happened otherwise. Here is one such interaction that I was involved with in the summer of 2021. I sat on my sofa or at my kitchen table, and became involved in a large and successful international collaboration.

### History

At the British Combinatorial Conference in 2009, Shamik Ghosh posed a problem about a certain graph defined on the set of elements of a finite group. He and I worked on the problem and published a paper, and later I was able to come up with a solution. (I will tell you later what the problem was.) This is not a subject I would normally work on, but Shamik’s problem had a certain attraction; and working on it had far-reaching consequences, some of which I am going to tell you about in this lecture.

### Kerala

In 2010, I was invited to an International Conference on Recent Trends in Graph Theory and Combinatorics in Kochi, Kerala, India, organised by Ambat Vijayakumar. This was a satellite meeting of the International Congress of Mathematicians held in Hyderabad that year. My conference talk was on a completely different topic, synchronization. After the conference, instead of going on to the ICM, I decided to take a holiday in Kerala. (On my first visit to India, in 1988, a group of students in Mumbai had told me that the parts of India I really must see were in the far north and far south, Kashmir and Kerala.)

### God’s Own Country

A few years later, four Iranian mathematicians, Ghodratallah Aalipour, Saieed Akbari, Reza Nikandish and Farzad Shaveisi, asked me a different question about this topic. We worked on it together and published a paper in 2017. At this point, I realized that I was chipping away at the edges of a much larger topic; so I stood back a bit to get it into perspective, and wrote a long survey paper. In the academic game, one is supposed to try to publish in the best possible journal, to score points for the department in the REF and to help one’s own promotion prospects. But I am getting too old for all that, so I did something different: I put it on the arXiv, and thought that was that.
### Alireza Abdollahi

The arXiv is a wonderful repository, where you can upload any scientific paper, and anyone can read it for free; and the research is really up-to-the-minute. The papers are not refereed, so if you read a paper, you have to make your own estimate of its value. Alireza Abdollahi is editor-in-chief of the *International Journal of Group Theory*, based in Isfahan, Iran. When he saw my paper on the arXiv, he wrote to me and invited me to submit it to his journal. Since the journal is “diamond open access” (free to both authors and readers), I happily agreed. I strongly believe that we should be publishing in journals like that! It has now appeared, in volume 11 (2022), 43–124.

### Ambat Vijayakumar

Someone else who saw the paper on the arXiv was Ambat Vijayakumar, who as you will recall had organised the 2010 conference in Kochi. He also wrote to me and suggested running a research discussion on the topic of the paper. He and his colleague Aparna Lakshmanan would do the organisation.

### The directed power graph

Back in 1955, Brauer and Fowler constructed the following graph from a group $G$. The vertices of the graph are the elements of $G$; we put an edge joining $x$ to $y$ if and only if $x$ and $y$ commute, that is, $xy = yx$. So this is the commuting graph of $G$.

(Some people, including Brauer and Fowler, remove the vertex corresponding to the identity element; this vertex would be joined to everything, which is not very interesting.) They used this graph to prove that there are only a finite number of finite simple groups which contain an involution (an element of order 2) whose centralizer (the set of elements which commute with it) has a prescribed structure. This was, arguably, the first step in the thousand-mile journey to the Classification of Finite Simple Groups, one of the great achievements of twentieth-century mathematics.

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### The power graph

As time went by, people began studying the commuting graph of a group in its own right, and found that it contained important information about the group. (For one example, a random walk on the commuting graph can be used to find a uniformly random conjugacy class of $G$, even though conjugacy classes can have many different sizes.)

In 1999, Kelarev and Quinn defined the power graph of a group $G$: again the vertex set is $G$, and $x$ and $y$ are joined if one is a power of the other (that is, $y = x^n$ or $x = y^n$ for some $n$). This graph also encodes information about the group, but Kelarev and Quinn seemed more motivated by finding graphs with good properties as networks.

### The directed power graph

The power graph is “really” a directed graph: put a directed edge from $x$ to $y$ if $y$ is a power of $x$. Ignoring the directions gives the power graph.

The question Shamik Ghosh asked me was, roughly, to what extent does the power graph determine the group? But another related question is, to what extent does the power graph determine the directed power graph? i.e. can we recover the arrows on the edges? The answer is, not completely, but we can up to isomorphism: two groups with isomorphic power graphs have isomorphic directed power graphs, even if the groups themselves are different. This holds only in the finite case: for infinite groups, it goes wrong, although something can be salvaged.

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Vijay advertised it, limiting places in the discussion to 50; it was way oversubscribed, by mathematicians mostly from India but also Iran, Japan and the USA. The discussion group would meet weekly, except that he invited me to give two talks in the first week to start things off. He didn’t know how long it would last, maybe a few weeks; such was the interest that it ran from May until August, when I gave a summary of some of the things we had achieved, on and off line.

So what was it about?

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Many other graphs have been considered. With Aalipour et al., I defined the enhanced power graph; with Bojan Kuzma, I defined the deep commuting graph. Also, two vertices $x$ and $y$ are joined in the commuting graph if and only if the group they generate is abelian. This suggests other graphs, where the rule is changed by putting another graph property in place of “abelian” (for group theorists, nice properties are “nilpotent” or “soluble”).

Earlier, the generating graph of a group had been defined, with $x$ and $y$ joined if they generate the whole group.

In the research discussion

A research discussion involving a fairly large group of mathematicians with various backgrounds, all pursuing their own interests, is perhaps not the best place to mount a team effort on an important but hard problem. But what it is good for, is bringing new ideas to the table. So Lavanya Selvaganesh talked about her work involving taking the power graph and shrinking the equivalence classes of a certain natural relation (having the same order) to a single vertex. We took up this question and extended it to other types of graphs and other equivalence relations.

Tamizh Chelvam pointed us to some similar graphs defined on rings rather than groups, such as zero divisor graphs. By asking questions about these similar to the ones we considered for groups, we found some new results.

Other types of graph, such as subgroup sum graphs and subspace inclusion graphs, were also considered in the discussion. Furthermore, Vikramin Arvind from Chennai and I began a new project trying to see just how much information about a group can be recovered from its commuting graph. There certainly are different groups with isomorphic commuting graphs, but we came up with a nice conjecture about when this can happen.

Also I continued a project with Ranjit Mehatari. There is an important class of graphs going by various names including cographs: this is the smallest class containing the 1-vertex graph and closed under taking complements and disjoint unions. The class of graphs for which the power graph is a cograph includes groups in which every element has prime power order (the so-called EPPO groups), and we are trying to determine all of these groups.

In a cyclic group of order $n$, there are $\phi(n)$ elements which generate the group, where $\phi(n)$ is Euler’s totient function, the number of integers $k$ with $0 \leq k < n$ and $\gcd(k,n) = 1$. Any two generators are joined in the power graph, so the clique number is at least $\phi(n)$. In fact it is a little larger, since every element of $C_n$ is joined to every generator. How much larger?

We define a number-theoretic function $F$ so that $F(n)$ is the clique number of the cyclic group $C_n$. Then we can write a recursion for $F$:

$$F(1) = 1, \quad F(n) = \phi(n) + F(n/p) \text{ for } n > 1,$$

where $p$ is the smallest prime divisor of $n$. Using this, we can show that $F(n) < 3\phi(n)$. In fact,

$$\lim sup F(n)/\phi(n) = 2.6481017597 \ldots$$

We have a formula for this mysterious constant but do not know much about it: for example, is it rational or irrational, algebraic or transcendental?
V. V. Swathi talked about the matching number of the power graph of a group (this is the maximum number of pairwise disjoint edges), and we subsequently continued this research. Although we have not yet managed to calculate the matching number for all groups, we were able to prove that the power graph and the enhanced power graph have the same matching number.

One very nice feature of the research discussion was that, unlike some St Andrews students, the participants were very ready to contribute to the discussion with questions, additions, or remarks. This is one of the main reasons why it worked so well.

By the time I gave the closing talk in the research discussion, it was clear that my survey article was well out of date; it would probably take a monograph to cover everything we know now. Since then, papers resulting from the discussions are beginning to appear. This will take a while, since there is a lot of material. I hope that this seminar has given many researchers a wealth of problems to study, confidence to study them, and tools for solving them.

And of course, there are plenty of interesting questions, which I am happy to suggest to anyone interested!

… for your attention.