Fifty years around Jan: Some history and some mathematics

Peter J. Cameron, University of St Andrews

Saxl Day
Cambridge, 22 July 2022
In 1968, I arrived in Britain, on the Shaw Savill liner Southern Cross, which docked in Southampton on 21 August. I spent most of the time between then and the start of term in Oxford as a tourist in London. There was a Soviet exhibition at the Commonwealth Institute, and I went along to have a look. I found the place surrounded by demonstrators protesting the Soviet invasion of Czechoslovakia. In fact the Russian tanks had rolled into Prague on 21 August. The same event brought Jan to Britain. He learned of the invasion while on a train returning home after a holiday here; he got off the train and returned to Britain. Both of us made our homes here.

Jan came to Oxford in 1970 to do his DPhil under Peter Neumann's supervision. This was the start of our fifty years of friendship. If I am not mistaken, I was his internal examiner. Later in the 1970s, Peter, Jan and I went to a finite geometry conference in the Isle of Thorns, the University of Sussex conference centre in Ashdown Forest. It was a beautiful early autumn, with berries in the hedges and golden leaves on the trees, so we decided to walk. The journey took us three and a half days, with overnight stops in Reading, Guildford, and East Grinstead.

On my last day in Caius, at the President's garden party, proceedings were interrupted by the Master for a little ceremony at which he formally admitted me as a member of the College.

The downside of my stay in Caius was that Jan was away on leave for most of the time, so the opportunity to work together was lost. But I am honoured to be a member of a college that had Jan as a fellow (as well as the other distinguished people I mentioned).

In 2008 I directed a six-month programme on Combinatorics and Statistical Mechanics at the Isaac Newton Institute. Jan generously arranged for me to be a by-fellow of Caius. I very much appreciated and enjoyed my time there; my only duty was to give three or four lectures to the undergraduate mathematicians. The titles of my four lectures were “Before and beyond Sudoku”, “Proving theorems in Tehran”, “Transgressing the boundaries”, and “Cameron felt like counting”.

Fellows of Caius, including John Venn, Ronald Fisher and John Conway, made appearances in the lectures.

In 1968, I arrived in Britain, on the Shaw Savill liner Southern Cross, which docked in Southampton on 21 August.

Jan came to Oxford in 1970 to do his DPhil under Peter Neumann's supervision. This was the start of our fifty years of friendship. If I am not mistaken, I was his internal examiner. Later in the 1970s, Peter, Jan and I went to a finite geometry conference in the Isle of Thorns, the University of Sussex conference centre in Ashdown Forest. It was a beautiful early autumn, with berries in the hedges and golden leaves on the trees, so we decided to walk. The journey took us three and a half days, with overnight stops in Reading, Guildford, and East Grinstead.

On my last day in Caius, at the President's garden party, proceedings were interrupted by the Master for a little ceremony at which he formally admitted me as a member of the College.

Both of us made our homes here.

The first and second were part of a long-standing obsession of mine: the action of permutation groups on unordered subsets of the domain. The third led on to my paper with Laci Babai showing that almost all primitive groups except for alternating groups are automorphism groups of edge-transitive hypergraphs. The fourth is of course the most celebrated of the four.

I wrote four joint papers with Jan:


The first and second were part of a long-standing obsession of mine: the action of permutation groups on unordered subsets of the domain. The third led on to my paper with Laci Babai showing that almost all primitive groups except for alternating groups are automorphism groups of edge-transitive hypergraphs. The fourth is of course the most celebrated of the four.

I don’t want to spend time on old results. Here are some very new things that I think Jan would have enjoyed, and I would have enjoyed explaining to him. The first concerns a theorem of Landau from 1903.

Theorem

Given a positive integer $k$, there are only finitely many finite groups with $k$ conjugacy classes.

Using graph theory we have found a strengthening of this theorem. (We in this case means Parthajit Bhowal, Rajat Kanti Nath, Benjamin Sambale and me.) The soluble conjugacy class graph of a finite group is the graph whose vertices are the conjugacy classes, two vertices $C$ and $D$ joined if there exist $g \in C$ and $h \in D$ such that $(g,h)$ is a soluble group.
The clique number of a finite graph is the size of the largest complete subgraph (set of vertices with every two joined by an edge).

Theorem
Given a positive integer k, there are only finitely many finite groups whose soluble conjugacy class graph has clique number k.

Our proof uses the Classification of Finite Simple Groups, but only in a light-touch way. I don’t know whether this can be avoided.

For Landau’s theorem, there are now explicit bounds. Such bounds for our theorem are not known: an open problem for somebody to tackle.

Some more mathematics
This is a result with Marina Anagnostopoulou-Merkouri, an undergraduate doing a research internship at St Andrews.

The philosophy is that, if we have a hierarchy of permutation group properties (say P and Q, where Q is stronger than P), we would like a property (which we call pre-Q) which is independent of P but such that P and pre-Q are equivalent to Q. For P and Q being quasi-primitivity and primitivity, this turns out to be a rich field to investigate.

A transitive permutation group $G$ on $\Omega$ is pre-primitive if every $G$-invariant partition of $\Omega$ is the orbit partition of some subgroup of $G$. We work is still in progress, but here are a few of our results.

Theorem
A permutation group is primitive if and only if it is quasiprimitive and pre-primitive.

Pre-primitivity is closed upwards.

A regular permutation group $G$ is pre-primitive if and only if it is a Dedekind group, that is, all subgroups are normal.

A wreath product of transitive groups (in its imprimitive action) is pre-primitive if and only if the two factors are pre-primitive.

We have also defined a property pre-synchronizing so that a permutation group $G$ is synchronizing if and only if it is primitive and pre-synchronizing.

However, this is not so interesting:

Theorem
A pre-synchronizing group is either primitive (and hence synchronizing) or else the Klein group of order 4 acting regularly. Perhaps there are other hierarchies of permutation group properties where similar ideas can be applied …

Farewell Jan