The ADE affair

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Calling card

What should humanity send as a calling card for extraterrestrial civilisations? The Pioneer engineers decided that this would do:

Francis Buekenhout had another idea:

A modern Hilbert problem?

In an article on “Problems of present day mathematics” in an AMS Symposium on Mathematical Developments arising from Hilbert Problems, V. Arnold wrote, The Coxeter–Dynkin graphs $A_n$, $D_n$, $E_k$ appear in many independent classification theorems. For instance:

- the platonic solids (or finite orthogonal groups in Euclidean 3-space).
- the categories of linear spaces and maps.
- the singularities of algebraic hypersurfaces with a definite intersection form of the neighboring smooth fiber.
- critical points of functions having no modules.
- the Coxeter groups generated by reflections, or of Weyl groups with roots of equal length.

Arnold forgot to mention one of the best-known occurrences, in the classification of simple Lie algebras and Lie groups. Since he wrote, several more such classifications have arisen, including

- graphs with least eigenvalue $-2$;
- instantons and asymptotically locally Euclidean spaces;
- cluster algebras;
- and this year, a proof that $E_8$ gives the densest possible sphere packing in 8 dimensions won a Fields medal for Maryna Viazovska.

Buekenhout’s view was that, even if an alien civilisation had very different mathematics to ours, chances are that they would have come up with at least some of the areas in which the ADE diagrams occur.

I will say a bit about some of this rich tapestry.

The extended diagrams

Closely related to the ADE diagrams are the so-called extended diagrams. In each case the extension adds one vertex, as follows:

- for $A_n$, the new vertex is joined to both ends of the path, forming an $(n + 1)$-cycle;
- for $D_n$, it creates a fork at the other end of the diagram, or a $K_{1,4}$ if $n = 4$;
- for $E_n$, it extends one of the arms, so that the numbers of vertices on the arms are $(3, 3, 3)$ $(n = 6)$, $(2, 4, 4)$ $(n = 7)$, or $(2, 3, 6)$ $(n = 8)$.

What are these diagrams?

The spectrum of a graph is the spectrum of its adjacency matrix. J. H. Smith proved the following theorem.

**Theorem**

- The connected graphs with greatest eigenvalue strictly less than 2 are precisely the ADE diagrams.
- The connected graphs with greatest eigenvalue equal to 2 are precisely the extended ADE diagrams.

Although this theorem was proved in 1969, it is in some sense implicit in the classification of simple Lie algebras over $\mathbb{C}$. 


Sketch proof

Here is an outline proof of the first part.
It is easy to see that the extended ADE diagrams have greatest eigenvalue 2, as we will see shortly. So a connected graph whose greatest eigenvalue is less than 2 cannot contain any of these.

In particular, it does not contain $\tilde{A}_n$ (a cycle), and so is a tree; it does not contain $\tilde{D}_n$, and so has at most one branchpoint, such a point having valency at most 3; and it does not contain $\tilde{E}_n$, and so the lengths of the three arms are restricted to the appropriate values.

Polyhedral and tessellations

The numbers of vertices on the arms of the $E_n$ diagrams are $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$ for $n = 6, 7, 8$ respectively. This should remind you of the regular polyhedra in 3-space.

The corresponding numbers for the extended diagrams are, as we saw, $(3, 3, 3)$, $(2, 4, 4)$ and $(2, 3, 6)$ respectively, corresponding to the regular tessellations of the Euclidean plane by triangles, squares and hexagons.

These are not accidental; we will return to them later.

A detour

By the Perron–Frobenius Theorem, a connected graph has a unique eigenvector (up to scalar multiple) corresponding to the greatest eigenvalue; this eigenvector has all its entries positive. This means that we can label the vertices of the extended ADE diagrams with positive numbers so that the sum of labels on the neighbours of a vertex $v$ is twice the label on $v$.

It is a simple exercise to write down such a labelling. See the next slide for $\tilde{E}_8$.

Finite rotation groups

As is well known, the list of finite groups of rotations in 3 dimensions (or finite subgroups of $SO_3(\mathbb{R})$) is as follows:

- cyclic groups $C_n = \langle x | x^n = 1 \rangle$;
- dihedral groups $D_n = \langle x, y | x^n = y^2 = (xy)^2 = 1 \rangle$;
- the tetrahedral group $\langle x, y : x^2 = y^3 = (xy)^3 = 1 \rangle$;
- the octahedral group $\langle x, y : x^2 = y^3 = (xy)^4 = 1 \rangle$;
- the icosahedral group $\langle x, y : x^5 = y^3 = (xy)^5 = 1 \rangle$.

The correspondence with the ADE diagrams is clear.

Binary rotation groups

There is a two-to-one homomorphism from the special unitary group $SU_2(\mathbb{C})$ to the rotation group $SO_3(\mathbb{R})$.

The inverse image of each finite rotation group $G \leq SO_3(\mathbb{R})$ is a double cover $\tilde{G}$ of $G$ in $SU_2(\mathbb{C})$, a group with a centre $Z$ of order 2 such that $G/Z = \tilde{G}$. (Note, incidentally, that $Z$ contains the unique involution in $\tilde{G}$.)

Each of these groups comes with a “natural” two-dimensional unitary representation $\rho$, which is in fact self-dual (implying that $\rho \otimes \rho$ contains the trivial representation).
McKay's observation

From the preceding, we see that there is a graph structure on the set of irreducible complex representations of \( G \): an edge joins the representations \( \sigma_1 \) and \( \sigma_2 \) if \( \sigma_2 \) is a constituent of the representation \( \sigma_1 \otimes \rho \). Our earlier observation shows that the graph is undirected. If we label each node with the degree of the corresponding representation \( \sigma \), we see that the labels of the neighbours of \( \sigma \) add up to the degree of \( \sigma \otimes \rho \), that is, twice the degree of \( \sigma \). So we have a positive eigenvector with eigenvalue 2. We conclude that the graph is an extended ADE diagram. Indeed, the correspondence agrees with the one we just observed for the rotation groups.

A story

When I was a fellow of Merton College, Oxford, Peter Kronheimer was an undergraduate at the college. In his final year, he came to ask me about McKay’s observation. After obtaining a very good First in his finals, he did a DPhil under the supervision of Michael Atiyah. In his thesis, he used this observation to construct a family of instantons, or asymptotically locally Euclidean (ALE) spacetimes. I am afraid I am not competent to tell you about this, but I am very happy to have had a part in the story!

Another story

The moral of the ADE classification is that, if you solve a classification problem to which the answer turns out to be “two infinite families and three sporadic examples”, then chances are that your problem is connected with the ADE affair. In the 1970s, two of my collaborators, Jaap Seidel and Jean-Marie Goethals, were working on graphs with smallest eigenvalue −2. I visited Jaap in Eindhoven; he explained to me that he had solved a technical problem that arose, and found one infinite family and three sporadic examples. The next day, we visited Jean-Marie in Brussels, who explained that he had also succeeded, and found one infinite family and three sporadic examples. On comparing their work, each of them had missed a different infinite family. So it was clearly an ADE problem, as I will now describe.

Root systems

A root system is a very geometric object. It is a finite set \( R \) of non-zero vectors in Euclidean space \( \mathbb{R}^n \) with the properties:

- If \( v \in R \), then for any real number \( c \), \( cv \in R \iff c = \pm 1 \).
- For all \( v, w \in R \), \( 2(v \cdot w)/(v \cdot v) \in \{0, \pm 1, \pm 2, \pm 3\} \).
- For all \( v \in R \), the reflection \( \sigma_v \) in the hyperplane perpendicular to \( v \) maps \( R \) to itself.

Note that:

- \( 2(v \cdot w)/(v \cdot v) \times 2(v \cdot w)/(w \cdot w) \leq 4 \) by Cauchy–Schwarz, so if there are roots of different lengths then the ratio of their squared lengths is 2 or 3.
- \( \sigma_v(w) = w - 2(v \cdot w)/(v \cdot v)v \), an integer linear combination of \( v \) and \( w \). From this, it can be shown that \( R \) spans a lattice (a root lattice).

Roots of constant length

It is clear that the vectors of fixed length in a root system form themselves a root system, so it is enough to classify these, and then piece together at most two such for the general classification.

If all the roots have the same length, then the angles between two roots are 60°, 90°, 120°, or 180°. It can be shown that there is a basis consisting of roots with non-positive inner products. If \( G \) is the Gram matrix, then (after normalisation) we have \( G = 2I - A \), where \( A \) is the adjacency matrix of a graph. Since \( G \) is positive definite, \( A \) has greatest eigenvalue less than 2. Thus, if the root system is indecomposable (that is, the graph is connected), it is an ADE diagram. Moreover, the graph determines the root system.

From the icosahedron to \( E_8 \)

Is there a direct connection between the 3-dimensional rotation groups and the root systems, underlying the McKay correspondence? A beautiful construction was found recently by Pierre Dechant, using Clifford algebras; but I cannot describe it here. I merely note one small piece of numerology: the number of roots in the \( E_8 \) root system (240) is twice the number of rotations and reflections of the icosahedron.
Star-closed sets

Let $A$ be the adjacency matrix of a graph with least eigenvalue $-2$. Then $A + 2I$ is positive semidefinite, and so is the matrix of inner products of a set of vectors in $\mathbb{R}^n$, where $n$ is the comultiplicity of $-2$ as an eigenvalue. Clearly any two of these vectors make an angle 90° or 60°. Now a geometric argument shows that a set of lines through the origin in $\mathbb{R}^n$, in which any two lines make angle 90° or 60°, and is maximal with respect to this property, is star-closed: that is, if two lines in the set make angle 60°, then the third line in their plane at 60° to both is also in the set. Taking vectors of fixed length on the lines in such a star-closed set, we obtain a root system!

The classification

Thus, graphs with least eigenvalue $-2$ are “contained” in a root system of type ADE. It can be shown that the $A_n$ root system is contained in $D_{n+1}$. Moreover, graphs represented in the root system $D_{n+1}$ are what Alan Hoffman defined as generalised line graphs. So we have

Theorem

A connected graph with smallest eigenvalue $-2$ (or greater) is either a generalized line graph, or one of finitely many exceptions (all these exceptions being represented in the root system $E_8$).

The paper containing this theorem (with Goethals, Seidel, and Ernest Shult) is one of my most cited.

And more . . .

I have not told you the whole story. There are connections with algebras of finite representation type, with critical points of smooth functions, and with cluster algebras (which come up in the theories of Poisson algebras and totally positive matrices). Indeed, when Fomin and Zelevinsky invented cluster algebras, they didn’t at first know that the finite dimensional ones fitted the ADE classification, but discovered this later.

Conclusion

So, have mathematicians succeeded in answering Arnold’s question, or have we simply added further layers of mystery?

You decide . . .

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<th>$A_n$</th>
<th>$C_n$</th>
<th>$D_n$</th>
<th>$E_8$</th>
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<tbody>
<tr>
<td>(n ≥ 3)</td>
<td>(n ≥ 3)</td>
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