Graphs on groups and their relations

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Graphs on groups

Probably to most people, “graphs on groups” means Cayley graphs. I am talking about something different, typefied by the commuting graph of a group; the vertices are the group elements, two vertices joined if they commute. There are several reasons why we might be interested in graphs like this.

1. The graph gives information about the group. The commuting graph was introduced by Brauer and Fowler in their seminal paper in 1955. They showed that, in a non-abelian finite simple group of even order, elements are not too distant in the graph, and used this to show that there are only finitely many such groups with a prescribed involution centralizer. This was perhaps the first step in the thousand-mile journey to the Classification of Finite Simple Groups.

2. The group gives us constructions of interesting graphs. For example, the power graph (two vertices joined if one is a power of the other) of the Mathieu group M_{11} contains within it some interesting graphs with large girth. This is probably true for other simple groups as well; exploration of this is underway.

3. The interaction between group and graph(s) enables us to define interesting classes of groups, and has led to new results in group theory and new characterisations of interesting classes of graphs.

There is far too much known on this topic for a complete account here. I will try to tell the story mostly by examples.

Personal history

A couple of years ago, I found myself thinking obsessively about these graphs, and so I wrote a 50-page survey article about them. Since I no longer have to strive to get papers in the best possible journals, I simply put it on the arXiv.

Two things happened as a result:

▶ Alireza Abdollahi invited me to publish it in the journal he edits, the International Journal of Group Theory. This is a diamond open access journal, so I was happy to accept.
▶ Ambat Vijayakumar and Arpana Lakshmanan at CUSAT in Kochi, Kerala, ran an on-line discussion group on graphs and groups, and invited me to be involved. This led to many new results and ideas.

A hierarchy

The two graphs I have mentioned, together with several others, form a hierarchy: each is contained in the next as a spanning subgraph. Here they are with the joining rules for elements \( g, h \in G \).

▶ The null graph.
▶ The power graph: one of \( g, h \) is a power of the other.
▶ The enhanced power graph: \( g, h \) generate a cyclic group.
▶ The deep commuting graph: the inverse images of \( g, h \) commute in every central extension of \( G \).
▶ The commuting graph: \( g, h \) generate an abelian group.
▶ The non-generating graph: \( g, h \) generate a proper subgroup of \( G \).
▶ The complete graph.

(In fact the commuting graph is contained in the non-generating graph provided \( G \) is either non-abelian or not 2-generated.)

When are two graphs equal?

Interesting classes of groups are defined by the condition that two graphs coincide. Some are trivial; for example, the power graph is null only for the trivial group, and the non-generating graph is complete only if \( G \) is not 2-generated.

▶ The power graph is equal to the enhanced power graph if and only if \( G \) is an EPPO group (all elements have prime power order). These groups were studied by Higman and Suzuki, and classified by Brandi in the early 1990s.
▶ The enhanced power graph is equal to the commuting graph if and only if all Sylow subgroups are cyclic or generalized quaternion. The classification follows from Burnside’s transfer theorem and the Gorenstein–Walter theorem.
▶ For \( G \) non-abelian, the commuting graph is equal to the non-generating graph if and only if \( G \) is a minimal non-abelian group. These groups were determined by Miller and Moreno in 1904.

The theory of the deep commuting graph was developed with Bojan Kuzma, and I will not speak about it here, though it is connected with interesting group theory (isoclinism, the Schur and Bogomolov multipliers, etc.)

All these graphs have the property that they are invariant under the automorphism group of \( G \). (This is not so obvious for the deep commuting graph, but Bojan and I prove it in our forthcoming paper.)

Other graphs could be added to the list, including the nilpotency graph \((g, h)\) joined if they generate a nilpotent group) and the solvability graph \((g, h)\) joined if they generate a solvable group.)
Into the second dimension

The hierarchy can be extended into a second dimension. A number of authors had studied special cases; the general case was considered by G. Arunkumar, Rajat Kanti Nath, Lavania Selvaganesh and me in a paper in press.

This involves choosing also a $G$-invariant equivalence relation. I will consider the relations of conjugacy and same order. Now given any graph type, say the power graph, we define the conjugacy superpower graph by the rule that $g$ and $h$ are joined if there exist $g'$ and $h'$ in the conjugacy classes of $g$ and $h$ respectively such that $g'$ and $h'$ are joined in the power graph. Similarly for other equivalence relations and other graph types.

When do two of these coincide?

Of the nine graphs in the picture, two of them (the order supercommuting graph and the order superenhanced power graph) coincide; the rest are all distinct in general. But equality defines more interesting classes:

- The conjugacy supercommuting graph of $G$ is equal to the commuting graph if and only if $G$ is a 2-Engel group, that is, satisfies the identity $[x,y,y] = 1$;
- the conjugacy superpower graph of $G$ is equal to the power graph if and only if $G$ is a Dedekind group, that is, one in which every subgroup is normal.

Comments

Dedekind groups are all known. Such a group is either abelian or of the form $A \times B \times C$ where $A$ is a quaternion group, $B$ an elementary abelian 2-group, and $C$ an abelian group of odd order.

Engel groups have had a lot of attention. Any nilpotent group of class 2 is 2-Engel, and every 2-Engel group is nilpotent of class at most 3 (shown independently by Hopkins and Levi). The first part uses the following result:

Theorem

A group $G$ satisfies the 2-Engel identity if and only if every centralizer is a normal subgroup.

The only proof we found in the literature was a StackExchange post by Korhonen, using a result of Kappe. Information on earlier proofs welcome!

Approximately equal?

We can widen the classes of groups by asking not for two graphs in the hierarchy to be equal but for them to have the same value for some monotone graph parameter. Here is an example.

Recall that the power graph and the enhanced power graph of $G$ are equal if and only if every element of $G$ has prime power order.

Theorem

- The power graph and enhanced power graph of $G$ have the same clique number if and only if the largest order of an element of $G$ is a prime power.
- For every finite group $G$, the power graph and enhanced power graph of $G$ have the same matching number.

There are many similar questions which could be asked!

Compressed supergraphs

We defined the supergraphs above to have vertex set the entire group, so that we could compare them with other graphs in the hierarchy. However, it would be more natural to define compressed versions of these graphs, in which we take the vertices to be the equivalence classes of the appropriate equivalence relation (conjugacy or same order, in my examples). Then two classes $C$ and $D$ are joined if there are vertices $x \in C$ and $y \in D$ which are joined in the original graph.

I will give you one example of this. We can use it to prove a strengthening of a classical result on finite groups.

In a finite group $G$, I will denote the conjugacy class of the element $g$ by $g^{cc} = \{ x^{-1}gx : x \in G \}$. I will consider the compressed conjugacy superpower graph for the relation of solvability. Thus, two classes $g^{cc}$ and $h^{cc}$ are joined in this graph if there exist $g' \in g^{cc}$ and $h' \in h^{cc}$ such that $(g',h')$ is a solvable group. This is known as the solvable conjugacy class graph, written $\Gamma_{sc}(G)$.

Landau’s theorem

In 1904, Landau showed:

**Theorem**

Given a positive integer \( k \), there are only finitely many finite groups which have \( k \) conjugacy classes.

I shall give the proof. By the Orbit-Stabilizer Theorem, the size of a conjugacy class is given by \( |g^C| = |G|/|C_G(g)| \), where \( C_G(g) \) is the centralizer of \( g \). Since \( G \) is the disjoint union of conjugacy classes, we have

\[
\sum_{i=1}^{k} \frac{1}{n_i} = 1,
\]

where \( n_i \) is the size of the centralizer of an element in the \( i \)-th conjugacy class.

Now it is an exercise to show that the above equation has only finitely many solutions; and the largest \( n_i \) is \( |C_G(1)| = |G| \).

A strengthening

With Parthajit Bhowal, Rajat Kanti Nath and Benjamin Sambale, I have proved:

**Theorem**

Given a positive integer \( k \), there are only finitely many finite groups \( G \) whose solvable conjugacy class graph has clique number \( k \).

The proof uses the Classification of Finite Simple Groups, but does not require very detailed knowledge about the groups. An important ingredient is a theorem of S. Dolfi, R. M. Guralnick, M. Herzog and C. E. Praeger, extending John Thompson’s N-group theorem: A finite group is solvable if and only if its solvable conjugacy class graph is complete.

Seeking the jewel in the lotus

To expand a compressed graph, we simply “blow up” each vertex to a complete graph on the equivalence class of the appropriate size.

We note that vertices in the same equivalence class have the same neighbours, that is, they are twins. (Formal definition shortly.)

In fact, all the graphs we have considered will have twins. For example, if the element \( g \in G \) has order \( m > 2 \), and \( \gcd(k, m) = 1 \), then \( g \) and \( g^k \) are twins in any of the graphs defined earlier.

Graphs with many pairs of twin vertices are not likely to be so interesting from some points of view. Since my aim in the rest of the talk is to produce beautiful graphs from groups, we will have to deal with the twins.

The cokernel may be trivial.

A **cograph** is a graph which has no induced subgraph isomorphic to the 4-vertex path \( P_4 \). This important class of graphs has many characterizations:

**Theorem**

The following conditions for a graph \( \Gamma \) are equivalent:

- \( \Gamma \) is a cograph.
- \( \Gamma \) can be built from 1-vertex graphs by the operations of disjoint union and complementation.
- The cokernel of \( \Gamma \) is the 1-vertex graph.

This raises the question: For which finite groups \( G \) is the power graph (or one of our other graphs) a cograph? Pallabi Manna will talk about this at the conference.

Twins and twin reduction

Two vertices \( v, w \) in a graph \( \Gamma \) are **twins** if they have the same neighbours except possibly for one another. We call them open or closed twins according as their open or closed neighbourhoods are equal.

The process of **twin reduction** consists of picking a pair of twins and identifying them, and continuing until no further twins remain.

**Theorem**

The result of applying twin reduction to a graph is unique up to isomorphism, independent of the order in which the reduction is done.

I will call the resulting graph the **cokernel** of \( \Gamma \).

Power graphs of simple groups

What follows is largely experimental. We take a finite simple group, remove the identity (which is joined to all other vertices), remove connected components which are complete, and then perform twin reduction.

It appears that non-abelian finite simple groups can be divided into three classes:

- The power graph is a cograph. These simple groups have been classified in a paper with Pallabi Manna and Ranjit Mehatari. We find certain groups \( \text{PSL}(2,q) \) and \( \text{Sz}(q) \) (the precise values of \( q \) depend on hard number-theoretic problems) and the group \( \text{PSL}(3,4) \).
- After the above process, we are left with a large number of isomorphic connected components. This occurs for \( A_7 \), \( \text{PSL}(2,23) \) and \( \text{PSL}(2,25) \). For example, \( A_7 \) gives a graph with 35 connected components, each isomorphic to a tree with three arms of length 3 radiating from a central vertex.
The graph that remains is connected, often with large girth, and nice structural properties, and with automorphism group equal to the automorphism group of the original simple group. This occurs for $\text{PSL}(3, 3)$, $\text{PSU}(3, 3)$ and $M_{11}$.

For these three simple groups, the number of vertices, diameter and girth are given in the table.

<table>
<thead>
<tr>
<th>Group</th>
<th>Vertices</th>
<th>Orbits</th>
<th>Diameter</th>
<th>Girth</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{PSL}(3, 3)$</td>
<td>754</td>
<td>4</td>
<td>11</td>
<td>12</td>
</tr>
<tr>
<td>$\text{PSU}(3, 3)$</td>
<td>784</td>
<td>7</td>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>$M_{11}$</td>
<td>1210</td>
<td>4</td>
<td>20</td>
<td>20</td>
</tr>
</tbody>
</table>

Relation to the hierarchy

It is easy to see that the independence graph is contained in the complement of the power graph, while the rank graph is contained in the complement of the enhanced power graph. In a preprint which has not yet appeared as far as I know, these two authors, with Colva Roney-Dougal and Saul Freedman, have determined the groups for which equality holds in either of these inclusions.

Variants of the generating graph

Recall the generating graph, where two elements are joined if they generate the group. Of course, if the group is not generated by two elements, this graph is null! Fortunately, all finite simple groups are 2-generated, but what about other groups?

Recently Andrea Lucchini and Daniele Nemmi made two definitions to get around this problem:

- The independence graph of $G$ joins $g$ to $h$ if $\{g, h\}$ is contained in a generating set for $G$ which is minimal (with respect to inclusion).
- The rank graph joins $g$ to $h$ if $\{g, h\}$ is contained in a generating set of minimum cardinality.

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