A bridge between algebra and combinatorics

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Peter Neumann memorial
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A preprint

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**Primitive permutation groups of degree $3p$**

by Peter M. Neumann.

This paper presents an analysis of primitive permutation groups of degree $3p$, where $p$ is a prime number, analogous to H. Wielandt's treatment (19) of groups of degree $2p$. It is also intended as an example of the systematic use of combinatorial methods as surveyed in §6 for distilling information about a permutation group from knowledge of the decomposition of its character. The work is organised into three parts. Part I contains the lesser half of the calculation, the determination of the decomposition of the permutation character. Part II contains a survey of the combinatorial methods and, based on these methods, the major part of the calculation. Part III ties up loose ends left earlier in the paper and gives a tabulation of detailed numerical results.
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In the 1960s, three streams previously separate began to converge, bringing different names, axioms, and techniques:

- In permutation group theory, the methods used by Schur and Wielandt on what are now called Schur rings were abstracted into combinatorial structures, largely by Donald Higman, who called them coherent configurations.
- In statistics, the underlying structures of partially balanced incomplete-block designs were abstracted into combinatorial structures, by R. C. Bose and his students, who called them association schemes.
- In the (then) Soviet Union, Boris Weisfeiler and his colleagues were attacking the graph isomorphism problem, and devised combinatorial structures which they called cellular rings.

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A coherent configuration is a collection $A_1, \ldots, A_r$ of square 0-1 matrices of the same size, summing to the all-1 matrix $J$ and having a subset which sums to the identity matrix $I$, closed under transposition, and having the property that for any $i, j$, we have

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I should stress that this definition can be given in terms of a colouring of the edges of the complete directed graph instead of matrices. Many combinatorial objects are special cases of coherent configurations. The definitions just given probably don’t conjure up a picture in your mind. So here is a special case.
Strongly regular graphs

A simple graph on \( n \) vertices is strongly regular if, for some integers \( k, \lambda, \mu \), it has the properties

- any vertex has \( k \) neighbours;

- any two adjacent vertices have \( \lambda \) common neighbours;

- any two non-adjacent vertices have \( \mu \) common neighbours.

An association scheme with \( r = 3 \) matrices is the same thing as a complementary pair of strongly regular graphs. The famous Petersen graph is an example, with \( k = 3, \lambda = 0, \mu = 1 \).
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In 1956, Helmut Wielandt proved that a finite primitive permutation group acting on a set $\Omega$ of size $2p$ (where $p$ is an odd prime) is 2-transitive, unless $p$ has the form $2a^2 + 2a + 1$ for some positive integer $a$, in which case it may have rank 3 (this means three orbits on the set $\Omega \times \Omega$, whose sizes are expressed in terms of the parameter $a$.)
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Peter Neumann’s aim was to prove a similar theorem for the case where $|\Omega| = 3p$, where $p$ is a prime greater than 3. Wielandt needed to do a lot of work decomposing the permutation character of his group, and then the combinatorial argument, though innovative, is fairly straightforward. For Neumann, on the other hand, the decomposition of the permutation character was easier, because of a theorem of Walter Feit proved in the meantime; but the combinatorial part is much more complicated, and the result too; there are three possible quadratic expressions for the prime $p$ as well as three sporadic values.
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So what now?

As I explained, Peter’s paper was never published. However, it is a tour de force, and had it been published it would have been recognised as an important link in the chain of ideas sketched earlier. I believe that it is too good to be lost.
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As noted, Wielandt first showed that the permutation character decomposes into irreducible constituents of degrees $1, p - 1,$ and $p$. From general theory, these numbers are the multiplicities of the eigenvalues of the matrices in the corresponding coherent configuration (these are the identity and the adjacency matrices of a strongly regular graph and its complement).
As noted, Wielandt first showed that the permutation character decomposes into irreducible constituents of degrees 1, $p - 1$, and $p$. From general theory, these numbers are the multiplicities of the eigenvalues of the matrices in the corresponding coherent configuration (these are the identity and the adjacency matrices of a strongly regular graph and its complement). In the case $p = 5$, the strongly regular graph is the famous Petersen graph, which we met earlier.
In fact the combinatorial part of Wielandt’s argument shows the following:

**Theorem**

Let $\Gamma$ be a strongly regular graph on $2n$ vertices, whose eigenvalues have multiplicities $1$, $n-1$ and $n$, for some natural number $n$. Then one of the following is true:

- $\Gamma$ or its complement is a disjoint union of $n$ edges;
- $\Gamma$ or its complement has the parameters found by Wielandt.

I am not sure who first noticed this. The proof is in my book with Jack van Lint. Note that in the second case, the Petersen graph and its complement are not the only examples; there are a number of further examples (the first pairs having $26$ vertices).
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