A bridge between algebra and combinatorics

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Πeter Neumann memorial
9 April 2022
Wielandt and Neumann

A coherent configuration is a collection $A_1, \ldots, A_r$ of square 0-1 matrices of the same size, summing to the all-1 matrix $I$ and having a subset which sums to the identity matrix $I$, closed under transposition, and having the property that for any $i, j$, we have

$$A_iA_j = \sum_{k=1}^{r} \sigma_k A_k.$$

If all the matrices are symmetric, it is an association scheme.

A cellular algebra was the same as a coherent configuration apart from a small difference. But the term “cellular algebra” has been used with a quite different meaning by Graham and Lehrer, so this term has dropped out of use.

I should stress that this definition can be given in terms of a permutation character, and then the combinatorial part of the definition can be given in terms of Cellular theory, instead of the original combinatorial part. This has been used by Graham and Lehrer to prove a number of important results.

The famous Petersen graph is an example, with $k = 3$, $\lambda = 0$, $\mu = 1$. A coherent configuration is a collection $A_1, \ldots, A_r$ of square 0-1 matrices of the same size, summing to the all-1 matrix $I$ and having a subset which sums to the identity matrix $I$, closed under transposition, and having the property that for any $i, j$, we have

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At about this time, another pioneer, Charles Sims, was investigating permutation groups using graph theory, specifically results of Bill Tutte. Whereas Higman and Neumann considered the coherent configuration associated with a permutation group, which takes all orbital graphs together and uses numerical and algebraic information, Sims chose a particular graph and went more deeply into its structure. This led him to his celebrated conjecture, later proved, using the Classification of Finite Simple Groups (CFSG) by three of Peter’s students together with Gary Seitz.

Since then, many parts of combinatorics, including designs, codes, and Latin squares, have been used in the study of permutation groups. In return, permutation group theory has contributed to several areas of combinatorics, including regular polytopes, fair games, and synchronizing automata. The two subjects are now close partners.

As I explained, Peter’s paper was never published. However, it is a tour de force, and had it been published it would have been recognised as an important link in the chain of ideas sketched earlier. I believe that it is too good to be lost. To explain why, I return to Wielandt’s proof. The first thing to note is that, since those far-off days, we have a new tool, CFSG, which can be used to show that the only case to arise in Wielandt’s theorem is $a = 1, p = 5$, in which case the group is the symmetric or alternating group of degree 5, acting on the ten 2-element subsets of a 5-set. A similar remark applies to Neumann’s theorem. So there is no reason to publish the paper as it is, since CFSG makes much stronger results possible. But that is not the end of the story …

As noted, Wielandt first showed that the permutation character decomposes into irreducible constituents of degrees 1, $p-1$, and $p$. From general theory, these numbers are the multiplicities of the eigenvalues of the matrices in the corresponding coherent configuration (these are the identity and the adjacency matrices of a strongly regular graph and its complement). In the case $p = 5$, the strongly regular graph is the famous Petersen graph, which we met earlier.

It is my belief that a similar but substantially more elaborate theorem is hiding in Peter’s calculations. This summer, a project student Marina Anagnostopoulou-Merkouri and I hope to work this out. In preparation for this, I have re-typed Peter’s paper in LaTeX from a smudgy scan of a photocopy, and I would be happy to send this to anyone interested. I should say that typing this paper out was a happy experience; it brought its author back vividly to my mind.

... for your attention.