Graphs defined on groups

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Algebraic graph theory is the area where these two very different subjects can meet and have a productive relationship.
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Before leaving the Brauer–Fowler paper, I will make two remarks.

As noted, they don't use the word graph, but they make extensive use of the graph distance, the length of the shortest sequence from $x$ to $y$ not containing the identity, where consecutive elements commute. Questions about connectedness and diameter of this graph now have an extensive literature.

The main result of the paper is that, given a group $H$ with a central involution, there are only finitely many finite simple groups having an involution whose centraliser is $H$. This result was fundamental to the Classification of Finite Simple Groups; their paper was perhaps the first step on this thousand-mile journey.
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This is not the complete *dramatis personae*, just the big stars. Some bit players will come in later. Indeed you can imagine some for yourself. Noting that $x$ and $y$ are joined in the commuting graph if and only if $\langle x, y \rangle$ is abelian, we could define a graph where the joining rule is $\langle x, y \rangle$ is nilpotent, or solvable, or ...
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My intention is to show that we gain something by considering these graphs together rather than individually. So I will mostly not present detailed results about a particular family. In order to get started, we observe that these graphs form a hierarchy; each is contained in the next as a spanning subgraph. This is the main reason for taking the vertex set in each case to be the whole group.
A hierarchy of graphs

The most interesting questions about the hierarchy of graphs concern their relations to one another: for which groups are two of the graphs equal? If not, what can we say about their difference?
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This innocent question leads to some deep and important group theory. For example, a paper in preparation by Freedman, Lucchini, Nemmi and Roney-Dougal (which I won’t have time to discuss).
The hierarchy

Here is the hierarchy, with notation and a brief reminder of the definition of adjacency of two elements $x$ and $y$. The vertex set is a group $G$ in each case.

- The null graph.
- The power graph $\text{Pow}(G)$: $x = y^m$ or $y = x^m$.
- The enhanced power graph $\text{EPow}(G)$: $\langle x, y \rangle$ is cyclic.
- The commuting graph $\text{Com}(G)$: $xy = yx$.
- The non-generating graph $\text{NGen}(G)$: $\langle x, y \rangle \neq G$.

Each is contained in the next, except that the commuting graph is contained in the non-generating graph if and only if $G$ is either non-abelian or has more than two generators (that is, for all groups except 2-generated abelian groups).
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The groups in each of these classes have been determined. Before explaining this, let me mention another graph associated with a finite group.
The Gruenberg–Kegel graph

The Gruenberg–Kegel graph, sometimes called the prime graph, of $G$ has vertices the prime divisors of $|G|$, with an edge joining $p$ and $q$ if $G$ contains an element of order $pq$. 

Gruenberg and Kegel showed that the augmentation ideal of the integral group ring of $G$ is decomposable if and only if this graph is disconnected. They gave a structural description of such groups in an unpublished manuscript; the result was later published by Gruenberg's student Williams.

**Theorem**

Let $G$ be a finite group whose Gruenberg–Kegel graph is disconnected. Then either

- $G$ is a Frobenius or $2$-Frobenius group; or
- $G$ is an extension of a nilpotent $\pi$-group by a simple group by a $\pi$-group, where $\pi$ is the set of primes in the connected component containing 2.
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EPPO groups

The group $G$ is an **EPPO group** ("Elements of Prime Power Order") if every element of $G$ has prime power order. These groups were studied by Higman in the 1950s; he determined the solvable ones. Following the discovery of his infinite family of simple groups, Suzuki was able to determine the simple EPPO groups. Subsequently Brandl gave a complete classification, which was rediscovered by several authors.
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- In the cyclic case, using Burnside’s transfer theorem, $G$ is metacyclic (i.e., has a cyclic normal subgroup with cyclic quotient).

- If the Sylow 2-subgroups are generalized quaternion, then using Glauberman’s $Z^*$-theorem and the Gorenstein–Walter theorem, $G$ has a normal subgroup $N$ of odd order; $G/N$ has a unique involution $z$, and the quotient by $\langle z \rangle$ is a known group.
Approximate equality?

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There are plenty of open questions here; the only case to have been looked at (as far as I know) is the power graph and enhanced power graph. Again not many results are known. Recall that these graphs are equal for a group $G$ if and only if every element of $G$ has prime power order.
Theorem

Let $\omega$ denote clique number, the size of the maximal complete subgraph. Then $\omega(\text{Pow}(G)) = \omega(\text{EPow}(G))$ if and only if the largest order of an element of $G$ is a prime power.
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Let $\mu$ denote matching number, the maximum number of pairwise disjoint edges. Then every finite group $G$ satisfies $\mu(\text{Pow}(G)) = \mu(\text{EPow}(G))$.

One slightly surprising thing about the second result is that we do not have a formula for the matching number of $\text{Pow}(G)$ for an arbitrary group $G$. The theorem is proved by showing that, given any matching in $\text{EPow}(G)$, we can find another matching of the same size which has fewer edges which don’t belong to $\text{Pow}(G)$.
An example

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**Theorem**

Let $q$ be a prime power, and $G = \text{PGL}(2, q)$. Then $\omega(\text{Pow}(G)) = \omega(\text{EPow}(G))$ if and only if one of

- $q$ is a Mersenne prime;
- $q + 1$ is a Fermat prime;
- $q = 8$.

For the largest order of an element of $G$ is $q + 1$. If $q$ and $q + 1$ are both proper powers, then $q = 8$, by the Catalan conjecture (proved fairly recently by Mihăilescu). Otherwise either $q$ or $q + 1$ is prime, giving the remaining cases.

So our problem includes the determination of all Fermat and Mersenne primes!
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Let \( q \) be a prime power, and \( G = \text{PGL}(2, q) \). Then \( \omega(\text{Pow}(G)) = \omega(\text{EPow}(G)) \) if and only if one of

- \( q \) is a Mersenne prime;
- \( q + 1 \) is a Fermat prime;
- \( q = 8 \).

For the largest order of an element of \( G \) is \( q + 1 \). If \( q \) and \( q + 1 \) are both proper powers, then \( q = 8 \), by the Catalan conjecture (proved fairly recently by Mihăilescu).
Otherwise either \( q \) or \( q + 1 \) is prime, giving the remaining cases. So our problem includes the determination of all Fermat and Mersenne primes!
Differences

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A class of finite graphs is universal if every finite graph can be embedded as induced subgraph in a graph in that class.
Universality

A class of finite graphs is universal if every finite graph can be embedded as induced subgraph in a graph in that class. The power graphs of finite groups do not form a universal class. For these graphs are comparability graphs of partial orders, and hence are perfect; in particular, they do not contain odd cycles of length greater than 3 or their complements. But this is the only restriction:
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**Theorem**

*If $\Gamma$ is the comparability graph of a finite partial order, then there is a finite group $G$ such that $\Gamma$ is isomorphic to an induced subgraph of $\text{Pow}(G)$.*
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**Theorem**

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But using our hierarchy, we can strengthen the last result.
Theorem
Suppose that the edges of a finite complete graph are coloured blue, yellow and white in any manner. Then the vertex set can be embedded into a finite group $G$ such that

- the blue edges belong to $\text{EPow}(G)$;
- the white edges belong to $\text{Com}(G)$ but not to $\text{EPow}(G)$;
- the yellow edges do not belong to $\text{Com}(G)$.

This gives us several universality results at once:

- ignoring the yellow-white distinction, enhanced power graphs form a universal class;
- ignoring the blue-white distinction, commuting graphs form a universal class;
- ignoring the blue-yellow distinction, the class of graphs of the form $(\text{Com} - \text{EPow})(G)$ is universal.
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A final topic

There is much much more that I haven’t talked about, and many many open problems. Please see the references, or email me if you want to discuss some of this or work on some open problems.
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I will finish with a topic from the sheaf of results that have been proved as a result of the research discussion group; this has some cute mathematics …
Clique number of the power graph

As a final topic, there is a sense in which the enhanced power graph is not much larger than the power graph. For example, while $\omega(Pow(G)) \leq \omega(EPow(G))$, it is true the $\omega(EPow(G))$ is bounded above by a function of $\omega(Pow(G))$. This can be seen by looking more closely at the clique number of $Pow(G)$. 
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In a cyclic group

Let $f(n)$ be the clique number of $\text{Pow}(C_n)$, where $C_n$ is the cyclic group of order $n$. 
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Let $f(n)$ be the clique number of $\text{Pow}(C_n)$, where $C_n$ is the cyclic group of order $n$. Then $f(n)$ is given by the recurrence

$$f(1) = 1; \quad \text{for } n > 1, \quad f(n) = \phi(n) + f(n/p),$$

where $\phi$ is Euler's totient function and $p$ is the smallest divisor of $n$. Hence $f(n)$ is bounded above by $3\phi(n)$. Hence $n$ is bounded above by $cm \log \log m$, where $m = f(n)$; and the same bound holds for the clique numbers $m$ and $n$ of the power graph and enhanced power graph of an arbitrary group.
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In fact,
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\lim \sup f(n) / \phi(n) = 2.6481017597 \ldots ,
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where the constant on the right is
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\sum_{k \geq 0} \prod_{i=1}^{k} \frac{1}{p_i - 1},
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where \(p_1, p_2, \ldots\) are the primes in order.
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\[ \text{▶ is this constant rational, algebraic or transcendental?} \]
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- is this constant rational, algebraic or transcendental?
- what other numbers are limit points of the set \(\{f(n) / \phi(n) : n \in \mathbb{N}\}\)?
References


Saul D. Freeman, Andrea Lucchini, Daniele Nemmi and Colva M. Roney-Dougal, Finite groups satisfying the independence property, in preparation.
Thank you . . .

. . . for your attention.