Permutation Groups and Transformation Semigroups
Lecture 1: Introduction

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Permutation groups

For any set $\Omega$, $\text{Sym}(\Omega)$ denotes the symmetric group of all permutations of $\Omega$, with the operation of composition. If $|\Omega| = n$, we write $\text{Sym}(\Omega)$ as $S_n$. We write permutations to the right of their argument, and compose from left to right: that is, $ag$ is the image of $a \in \Omega$ under the permutation $g \in \text{Sym}(\Omega)$, and

$$a(g_1g_2) = (ag_1)g_2.$$ 

A permutation group on $\Omega$ is a subgroup of $\text{Sym}(\Omega)$. An action of a group $G$ on $\Omega$ is a homomorphism from $G$ to $\text{Sym}(\Omega)$; its image is a permutation group on $\Omega$. Whenever we define a property of a permutation group, we use the name for a property of the group action.

An example

Let $G$ be the group of automorphisms of the cube, acting on the set $\Omega$ of vertices, edges and faces of the cube: $|\Omega| = 26$. The action is faithful, so $G$ is a permutation group. Automorphism groups of mathematical objects provide a rich supply of permutation groups. These objects can be of almost any kind.

Orbits and transitivity

Let $G$ be a permutation group on $\Omega$. Define a relation $\sim$ on $\Omega$ by the rule

$$a \sim \beta \text{ if and only if there exists } g \in G \text{ such that } ag = \beta.$$ 

$\sim$ is an equivalence relation on $\Omega$. (The reflexive, symmetric and transitive laws correspond to the identity, inverse, and closure properties of $G$.) The equivalence classes are called orbits; the group $G$ is transitive if there is just one orbit. Thus, a permutation group has a transitive action on each of its orbits.

In the example, there are three orbits: the 8 vertices, the 12 edges, and the 6 faces.

Another way to say this

There is another way to describe transitivity, which will be useful for further properties. We say that a mathematical structure built on the set $\Omega$ is trivial if it is invariant under $\text{Sym}(\Omega)$, and non-trivial otherwise. Thus,

- a subset of $\Omega$ is trivial if and only if it is either $\Omega$ or the empty set;
- a partition of $\Omega$ is trivial if and only if either it has a single part, or all parts are singletons (sets of size 1);
- a simple graph on $\Omega$ is trivial if and only if it is either the complete graph or the null graph.

So we can say:

*A permutation group $G$ on $\Omega$ is transitive if and only if there are no non-trivial $G$-invariant subsets.*

Transitive actions

Let $G$ act on $\Omega$, and take $a \in \Omega$. The stabiliser of $a$ in $G$ is the set

$$\{g \in G : ag = a\}.$$ 

It is a subgroup of $G$.

If $H$ is any subgroup of $G$, the (right) coset space of $H$ in $G$ is the set $G : H$ of right cosets $Hx$ of $H$ in $G$. There is a transitive action of $G$ on $G : H$, given by the rule

$$(Hx)g = H(xg).$$ 

Now there is a notion of isomorphism of group actions, and the following theorem holds:

Theorem

- Any transitive action of $G$ on $\Omega$ is isomorphic to the action of $G$ on the coset space $G : G_\alpha$, for $\alpha \in \Omega$.
- The actions of $G$ on coset spaces $G : H$ and $G : K$ are isomorphic if and only if $H$ and $K$ are conjugate subgroups of $G$.

Regular permutation groups and Cayley’s Theorem

A permutation group $G$ is regular on $\Omega$ if it is transitive and the stabiliser of a point is the identity subgroup. The right cosets of the identity are naturally in bijection with the elements of $G$. So we can identify $\Omega$ with $G$ so that the action of $G$ is on itself by right multiplication. Thus we have Cayley’s Theorem:

Theorem

Every group of order $n$ is isomorphic to a subgroup of $S_n$.

In particular we see that asking a group $G$ to be a transitive permutation group is no restriction on the abstract structure of $G$.
**Primitivity**

A transitive permutation group \( G \) on \( \Omega \) is **primitive** if the only non-trivial \( G \)-invariant partitions are the trivial ones (the partition with one part and the partition into singletons).

This can be said another way. A **block of imprimitivity** is a subset \( B \) of \( \Omega \) with the property that, for all \( g \in G \), either \( Bg = B \) or \( Bg \cap B = \emptyset \). Then \( G \) is primitive if and only if the only blocks of imprimitivity are \( \Omega \), singletons, and the empty set.

Consider our example \( G \), in its transitive action on the vertices of the cube. We see that \( G \) is imprimitive; indeed it preserves two non-trivial partitions:

- the partition into pairs of antipodal points (opposite ends of long diagonals);
- the partition into the vertex sets of two interlocking tetrahedra.

**Basic groups**

A **Cartesian structure** on \( \Omega \) is an identification of \( \Omega \) with \( A^d \), where \( A \) is some set. We can regard \( A \) as an “alphabet”, and \( A^d \) as the set of all words of length \( d \) over the alphabet \( A \). Then \( A^d \) is a metric space, with the Hamming metric (used in the theory of error-correcting codes); the distance between two words is the number of positions in which they differ.

A Cartesian structure is non-trivial if \( |A| > 1 \) and \( d > 1 \). Let \( G \) be a primitive permutation group on \( \Omega \). We say that \( G \) is **basic** if it preserves no non-trivial Cartesian structure on \( \Omega \).

Although this concept is only defined for primitive groups, we see that the imprimitive group we met earlier, the symmetry group of the cube acting on the vertices, does preserve a Cartesian structure. The automorphism group of a Cartesian structure over an alphabet of size 2 is necessarily imprimitive – generalise our argument for the cube to see this.

**Multiple transitivity**

If \( G \) acts on \( \Omega \), then it has induced actions on the set of \( t \)-element subsets of \( \Omega \), or the set of \( t \)-tuples of distinct elements of \( \Omega \), where \( t \leq |\Omega| \).

We say that \( G \) is **\( t \)-homogeneous** if the first action above is transitive, and **\( t \)-transitive** if the second is.

A \( t \)-transitive group is \( t \)-homogeneous. The symmetric group \( S_t \) is \( t \)-transitive for all \( t \leq n \), while the **alternating group** \( A_n \) is \( t \)-transitive for \( t \leq n - 2 \).

A \( 2 \)-homogeneous group is primitive. (Exercise; proof later.)

For \( t = 2 \), these properties have graph-theoretic interpretations:

- \( G \) is \( 2 \)-homogeneous if there are no non-trivial \( G \)-invariant undirected graphs on \( \Omega \);
- \( G \) is \( 2 \)-transitive if and only if there are no non-trivial \( G \)-invariant directed graphs on \( \Omega \).

**The O’Nan–Scott Theorem**

A permutation group \( G \) is called

- **affine** if it acts on a vector space \( V \) and its elements are products of translations and invertible linear transformations of \( V \), so that \( G \) contains all the translations;
- **almost simple** if \( T \leq G \leq \text{Aut}(T) \), where \( T \) is a non-abelian finite simple group, and \( \text{Aut}(T) \) its automorphism group (where \( T \) embeds into \( \text{Aut}(T) \) as the group of inner automorphisms or conjugations).

I won’t define **diagonal** groups; here’s an example. Let \( T \) be a finite simple group. Then \( T \times T \), acting on \( T \) by the rule

\[ x(y, h) = g^{-1}xyh \quad \text{for all } x, y, h \in G, \]

is a diagonal group. (The stabiliser of the identity is the diagonal subgroup \( \{(g, g) : g \in G\} \) of \( G \times G \).)

**Theorem**

Let \( G \) be a finite basic primitive permutation group. Then \( G \) is affine, diagonal, or almost simple.

**The Classification of Finite Simple Groups**

A non-identity group is **simple** if its only normal subgroups are itself and the identity subgroup.

The **Classification of Finite Simple Groups**, or CFSG, does what its name suggests:

**Theorem**

A finite simple group is one of the following:

- a cyclic group of prime order;
- an alternating group \( A_n \), for \( n \geq 5 \);
- a group of Lie type;
- one of 26 sporadic groups.

This theorem has revolutionised finite permutation group theory. I will end with one of its consequences.
Multiply transitive groups

**Theorem (CFSG)**
All finite 2-transitive groups are explicitly known.

**Corollary (CFSG)**
The only finite 6-transitive groups are the symmetric and alternating groups.
Indeed, there are only two 5-transitive groups which are not symmetric or alternating, the Mathieu groups $M_{12}$ and $M_{24}$ and only two further 4-transitive groups, the Mathieu groups $M_{11}$ and $M_{23}$.

Mind the gap between semigroups and groups!

To any semigroup we can add an identity to produce a monoid of size one larger. Nothing like this is possible for groups!

Note that the numbers of $n$-element semigroups and $(n + 1)$-element monoids are fairly close; this is because we can add an identity to an $n$-element semigroup to form an $(n + 1)$-element monoid. But numbers of groups are much smaller; the group axioms are much tighter!

Transformation semigroups
We recall the definitions.

- A semigroup is a set $S$ with a binary operation $\circ$ satisfying the associative law:
  \[ a \circ (b \circ c) = (a \circ b) \circ c \]
  for all $a, b, c \in S$.
- A monoid is a semigroup with an identity $1$, an element satisfying
  \[ a \circ 1 = 1 \circ a = a \]
  for all $a \in S$.
- A group is a monoid with inverses, that is, for all $a \in S$ there exists $b \in S$ such that
  \[ a \circ b = b \circ a = 1. \]

From now on we will write the operation as juxtaposition, that is, write $ab$ instead of $a \circ b$, and $a^{-1}$ for the inverse of $a$.

Two analogues of Sym($\Omega$)

For a set $\Omega$, let $T(\Omega)$ be the set of all the maps from $\Omega$ to itself, with the operation of composition. If $|\Omega| = n$, we write $T(\Omega)$ as $T_n$. Note that $T(\Omega)$ is a monoid; it contains Sym($\Omega$), and $T(\Omega) \setminus \text{Sym}(\Omega)$ is a semigroup. $T(\Omega)$ is the full transformation semigroup on $\Omega$.

The order of $T_n$ is $n^n$.
Also let $I(\Omega)$ denote the set of all partial bijections on $\Omega$ (bijections between subsets of $\Omega$), with composition ‘where possible’: if $f_i$ has domain $A_i$ for $i = 1, 2$, then $f_1 f_2$ has domain $(A_1 \cap A_2) f_1^{-1}$ and range $(A_1 \cap A_2) f_2$. Again, if $|\Omega| = n$, we write $I_n$. This is the symmetric inverse semigroup.

The order of $I_n$ is $\sum_{k=0}^{n} \binom{n}{k} k!$; there is no closed form for this expression.

Regularity
An element $a$ of a semigroup $S$ is regular if there exists $x \in S$ such that $axa = a$. The semigroup $S$ is regular if all its elements are regular. Note that a group is regular, since we may choose $x = a^{-1}$. The semigroup $T_n$ is regular (exercise).

Regularity is equivalent to a condition which appears formally to be stronger:

**Proposition**
If $a \in S$ is regular, then there exists $b \in S$ such that $aba = a$ and $bab = b$.

**Proof.**
Choose $x$ such that $axa = a$, and set $b = xax$. Then
\[
aba = axaxa = axa = a,
\]
\[
bab = xaxaxa = xaxa = xax = b.
\]

Idempotents
An idempotent in a semigroup $S$ is an element $e$ such that $e^2 = e$. Note that, if $axa = a$, then $ax$ and $xa$ are idempotents. In a group, there is a unique idempotent, the identity. By contrast, it is possible for a non-trivial semigroup to be generated by its idempotents.

**Proposition**
Let $S$ be a finite semigroup, and $a \in S$. Then some power of $a$ is an idempotent.

**Proof.**
Since $S$ is finite, the powers of $a$ are not all distinct: suppose that $a^m = a^{m+r}$ for some $m, r > 0$. Then $a^{m-i} = a^{m+r}$ for all $i \geq m$ and $i \geq 1$; choosing $i$ to be a multiple of $r$ which is at least $m$, we see that $a^m = a^{m+r}$, so $a^m$ is an idempotent.

It follows that a finite monoid with a unique idempotent is a group. For the unique idempotent is the identity, and, if $a' = 1$, then $a$ has an inverse, namely $a^{-1}$. 

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Inverse semigroups

The semigroup $S$ is an inverse semigroup if for each $a \in S$ there exists a unique $b \in S$ such that $aba = a$ and $bab = b$. We say that $b$ is the (von Neumann) inverse of $a$.

The symmetric inverse semigroup $I(\Omega)$ is an inverse semigroup. In an inverse semigroup, the idempotents commute, and they form a semilattice under the order relation $e \leq f$ if $ef = fe = f$. In $I(\Omega)$, the semilattice of idempotents is isomorphic to the Boolean lattice of all subsets of $\Omega$.

Permutation groups and transformation semigroups

Let $S$ be a transformation semigroup whose intersection with the symmetric group is a permutation group $G$. How do properties of $G$ influence properties of $S$. In particular, what can we say if $S = \langle G, a \rangle$ for some non-permutation $a$?

Here is a sample theorem due to Araújo, Mitchell and Schneider.

Theorem

Let $G$ be a permutation group on $\Omega$, with $|\Omega| = n$. Suppose that, for any map $f$ on $\Omega$ which is not a permutation, the semigroup $\langle G, f \rangle$ is regular. Then either $G$ is the symmetric or alternating group on $\Omega$, or one of the following occurs:

- $n = 5$, $G = C_5$, $C_3 \rtimes C_2$, or $C_5 \rtimes C_4$;
- $n = 6$, $G = PSL(2, 5)$ or PGL(2, 5);
- $n = 7$, $G = AGL(1, 7)$;
- $n = 8$, $G = PGL(2, 7)$;
- $n = 9$, $G = PGL(2, 8)$ or PTL(2, 8).

Basics of transformation semigroups

Any map $f : \Omega \to \Omega$ has an image

$$Im(f) = \{xf : x \in \Omega\},$$

and a kernel, the equivalence relation $\equiv_f$ defined by

$$x \equiv_f y \iff xf = yf,$$

or the corresponding partition of $\Omega$. (We usually refer to the partition when we speak about the kernel of $f$, which is denoted $Ker(f)$.) The rank $\operatorname{rank}(f)$ of $f$ is the cardinality of the image, or the number of parts of the kernel. Under composition, we clearly have

$$\operatorname{rank}(f_2f_1) \leq \min\{\operatorname{rank}(f_1), \operatorname{rank}(f_2)\},$$

and so the set $S_m = \{f \in S : \operatorname{rank}(f) \leq m\}$ of elements of a transformation semigroup which have rank at most $m$ is itself a transformation semigroup.

Idempotents in transformation semigroups

Suppose that $f_1$ and $f_2$ are transformations of rank $r$. The rank of $f_2f_1$ is at most $r$. Equality holds if and only if $\operatorname{Im}(f_1)$ is a transversal for $Ker(f_2)$, in the sense that it contains exactly one point from each part of the partition $Ker(f_2)$. This combinatorial relation between subsets and partitions is crucial for what follows. Here is one simple consequence.

Proposition

Let $f$ be a transformation of $\Omega$, and suppose that $\operatorname{Im}(f)$ is a transversal for $Ker(f)$. Then some power of $f$ is an idempotent with rank equal to that of $f$.

For the restriction of $f$ to its image is a permutation, and some power of this permutation is the identity.

Analogues of Cayley’s Theorem

Theorem

An $n$-element semigroup is isomorphic to a sub-semigroup of $T_{n+1}$.

In Cayley’s theorem, we let the group act as the group of right multiplications of itself. For a semigroup, this action may not be faithful. So first we add an identity $e$ to form a monoid. Now $ae = eb$ implies $a = b$ and all is well.

A similar but slightly harder theorem holds for inverse semigroups:

Theorem (Vagner–Preston Theorem)

An $n$-element inverse semigroup is isomorphic to a sub-semigroup of $I_n$.