Infinity: Cantor's paradise

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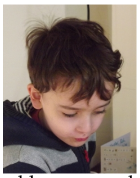
Is infinity a number?

Last month, my grandson Lex asked me, "Is infinity a number?"



When children compete to name larger and larger numbers, sooner or later someone will say "infinity". Is that a valid answer, and if so, is it the largest possible number (and so bring the contest to a halt)?
It turns out that this is the key question in the modern mathematical approach to infinity.

What should numbers do?

The things we do with numbers include

- counting (with whole numbers) or measuring (with real numbers);
- comparing magnitudes;
- adding and multiplying;
- subtracting and dividing.

If there are infinite numbers, they should be usable for at least some of these things.

Counting

> "I count a lot of things that there's no need to count," Cameron said. " Just because that's the way I am. But I count all the things that need to be counted."
>
> Richard Brautigan, *The Hawkline Monster: A Gothic Western*, Picador, 1976.

The really fundamental thing about numbers is that we use them to count things. Counting is more basic than other properties of numbers such as arithmetic or comparison.
When we count the apples in a bowl, or the Rollright Stones [though legend says these can't be counted!], we match them up with successive numbers: "One, two, three, . . . ".

Counting = Matching up

Georges Ifrah, in *From One to Zero*, tells of an American archaeological team excavating the palace of Nuzi, near Kirkuk in Iraq. They found a clay envelope inscribed with a list of 48 sheep and goats; inside there were 48 clay balls, which presumably the innumerate shepherd would use to check that his flock was complete.
The significance of the find was brought home when their uneducated servant, sent to the market to buy chickens, was unable to say how many chickens he had bought, but produced a collection of pebbles, one for each chicken.

We say that two sets have the same cardinality if they can be matched up. (**Note:** we haven't said yet what cardinality is!)

Cardinal numbers

In the days when physics was less sophisticated, the standard metre and the standard kilogram were defined by physical objects (the distance between two marks on a rod, and a lump of metal, respectively, made of platinum alloy). Other distances or masses could be measured by comparing them with the standard.
In the same way, we would like to count sets by having a standard number for each size of set, with which other sets can be compared. These standard numbers are called cardinal numbers.

The cardinal number for five-element sets should be a standard 5-element set whose definition does not involve the number 5 (to avoid paradoxes). For good logical reasons we take it to be the set $\{0, 1, 2, 3, 4\}$, the five numbers which precede 5.
There is a story that the logician Alfred Tarski, going on holiday, looked at his luggage and said, "Zero, one, two, three, four – good, I have all my five suitcases."
The added advantage is that this gives us a way to *construct* the natural numbers: each number is defined to be the standard set consisting of all its predecessors.

When we reach the infinite, we have a standard set

$$\{0, 1, 2, 3, 4, \ldots\}$$

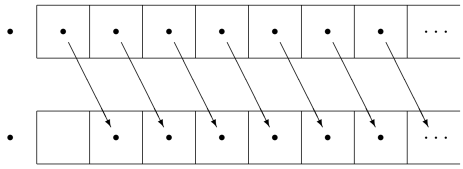
consisting of all the natural numbers. A set which can be matched up with this set is called countable (because we can count its elements with the natural numbers).
Cantor introduced the notation $\aleph_0$ (aleph-zero – “aleph” is the first letter of the Hebrew alphabet) for this set.
The next number should have the standard set

$$\{0, 1, 2, 3, 4, \ldots, \aleph_0\}$$

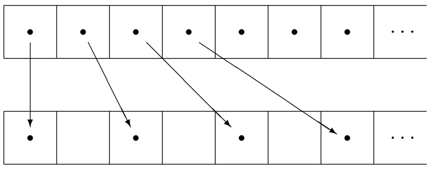
for sets obtained by adding one new element to an infinite set.
But things are not so simple ...

Hilbert’s Hotel

Hilbert described the first of Cantor’s discoveries in terms of an infinite hotel (which we now call Hilbert’s Hotel) with $\aleph_0$ rooms, numbered $0, 1, 2, \ldots$.
One night, the hotel is completely full, when a new guest arrives. What happens?
In a finite hotel, the guest would simply be turned away. But in Hilbert’s hotel, the manager simply asks the guest in room 0 to move to room 1, the guest in room 1 to move to room 2, and in general room n to room $n + 1$. Everyone still has a room, but now room 0 is free for the new guest.



So infinity plus one is infinity; we don’t need a new number.
If infinitely many guests arrive, we move the guest in room 0 to room 0, room 1 to room 2, and in general room n to room $2n$.
Then the even numbered rooms are occupied, but the odd numbered rooms are all free for the new guests:



So infinity plus infinity is still infinity.

Galileo revisited

This should remind you of Galileo’s argument. Cantor boldly went beyond the point where Galileo drew back; according to Cantor the set of perfect squares and the set of all natural numbers have the same cardinality, namely $\aleph_0$.
Cantor also showed that infinity times infinity is still infinity, and hence the paradoxical result that the number of rational numbers (or fractions) is the same as the number of natural numbers, even though the rational numbers lie densely on the line while the natural numbers are spaced out evenly.
Is there a larger infinity?

A larger infinity

Cantor’s most remarkable discovery was that there is a larger infinity: there are more real numbers (numbers represented by infinite decimals) than natural numbers.
The proof technique has become a classic: Cantor’s diagonal argument.
Consider what we have to do to prove this. We have to show that, no matter how we try, we cannot match up the real numbers with the natural numbers. In other words, if someone comes along with a list of real numbers $(x_1, x_2, x_3, \ldots)$, then the list cannot contain all real numbers: we must be able to show them a number which is not on the list. This is how it is done.
For simplicity we just consider real numbers between 0 and 1.
Even these are too numerous to match with the natural numbers.

Cantor’s diagonal argument

Suppose the list begins like this:

$$\begin{aligned} x_1 &= 0.67473\ldots \\ x_2 &= 0.38594\ldots \\ x_3 &= 0.22222\ldots \\ x_4 &= 0.45831\ldots \end{aligned}$$

and so on.
Consider the “diagonal” digits in the list. Write down the number formed by these:

$$y = 0.6823\ldots$$

Now change every 8 into a 5, and all other digits into 8:

$$z = 0.8588\ldots$$

This number z is not in the original list. For if $z = x_n$, then the nth digit of z would be the same as that of x_n; but this isn't so, since the nth digit of y was the same as that of x_n, and we changed it!
So the list was not complete.
Cantor denoted the cardinality of the set of real numbers by $\mathfrak{c}$.
So we can say that $\mathfrak{c} > \aleph_0$.

The Continuum Hypothesis

Cantor asked: Is $\mathfrak{c} = \aleph_1$? In other words, is the cardinality of the real numbers the next largest after the natural numbers, or could there be a subset of the real numbers of intermediate size?
Hilbert included this in his list of 23 problems to guide the development of mathematics in the 20th century.
The answer is not what either Cantor or Hilbert expected.

Independence of CH

The standard axioms for set theory are the Zermelo–Fraenkel axioms ZFC (including the so-called Axiom of Choice).
Hilbert's goal was to prove or disprove the Continuum Hypothesis in the system ZFC.
In 1940, Gödel showed that CH could not be disproved in ZFC, by building a model in which it holds.
In 1963, Cohen showed that it couldn't be proved in ZFC, by building a model in which it fails.
So we are left with a situation like that for Euclidean or non-Euclidean geometry: either CH or not-CH is a permissible assumption for a universe of set theory in which to do mathematics!

Larger and larger

Cantor also discovered that the cardinal numbers grow without limit, by an argument which "puts Russell's paradox to creative use".
He showed that given any set X whatever, the power set of X (the collection of all subsets of X) is larger.
For suppose we could match up each subset of X with an element of X. Then the subsets could be written as Y_a, for a running through the elements of X.
Let $Z = \{a \in X : a \notin Y_a\}$, the set of elements not lying in the set they label. By assumption, $Z = Y_b$ for some b. Now ask, is $b \in Y_b$? If it is, then it isn't; and if it isn't, then it is.
So no such labelling can exist.

Subtracting and dividing

> Counting is a less precise tool for infinite sets than for finite ones. The shepherdess who can count her flock of a hundred sheep will know if the wolf has taken one; but, if she has an infinite flock, she wont notice until almost all of her sheep have been lost.
>
> Peter J. Cameron, *Combinatorics*

In other words, $\aleph_0 - \aleph_0$ could be any natural number from 0 onwards, or it could be $\aleph_0$. So subtracting infinite numbers (and even more so, dividing them) doesn't make a lot of sense.
This is not really a problem: we already forbid division by zero!

Other systems

There is much I haven't discussed: for example,

- very large cardinal numbers (such as inaccessible cardinals, Ramsey cardinals, Erdős cardinals, ...);
- ordinal numbers (a number system in which infinity plus one really is larger than infinity);
- surreal numbers, a system in which infinity minus one makes sense (and there are infinitesimals as well).

But I think that is enough of this infinite topic...