A quick tour of algebraic graph theory

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Algebraic graph theory

Algebraic graph theory falls roughly into two parts:

- **Linear algebra**: properties of matrices associated with a graph, especially its eigenvalues, and how they give information about the graph;
- **Group theory**: automorphisms of graphs and constructions of graphs from groups.

I will take you on a quick tour through these topics and the connections between them. Since Stephan Wagner will be talking about spectral graph theory on Thursday, I will spend more time on the second part.

Graph isomorphism

If two graphs are isomorphic, then any graph-theoretic property of one holds also for the other, and any graph-theoretic parameter takes the same value on both graphs. So it is important to be able to check this. One of the big open problems in computational complexity asks:

**Problem**

*Can you decide in polynomial time whether two graphs are isomorphic?*

The last decade saw a major breakthrough by László Babai, who gave an algorithm for graph isomorphism that runs in quasi-polynomial time, that is, time $O(\exp(a \log n)^c)$ for some constants $a$ and $c$. (This is polynomial if $c = 1$.) However, for graphs of manageable size, practical algorithms such as Brendan McKay’s `nauty` out-perform Babai’s algorithm.

In matrix terms . . .

Let $\Gamma$ be a graph with $n$ vertices. Number the vertices from 1 to $n$, and define the adjacency matrix $A = A(\Gamma)$ to be the $n \times n$ matrix with $(i,j)$ entry 1 if the $i$th and $j$th vertices are adjacent, and 0 if not. Then $A$ is a real symmetric matrix, so there is an orthogonal (and so is orthogonal) direct sum of eigenspaces of $A$. If we re-number the vertices, by applying a permutation to the labels, then the new adjacency matrix $B$ satisfies $B = QAQ^T$, where $Q$ is a permutation matrix (and so is orthogonal); thus $B$ has the same eigenvalues and multiplicities as $A$. Thus, isomorphic graphs have the same eigenvalues and multiplicities.

Generalized line graphs

In the late 1970s, with Jean-Marie Goethals, Jaap Seidel and Ernie Shult, I proved a conjecture of Hoffman, which I now describe.

The **line graph** of a graph $\Gamma$ is the graph $L(\Gamma)$ whose vertices are the edges of $\Gamma$, with two vertices of $L(\Gamma)$ adjacent if and only if the corresponding edges of $\Gamma$ intersect.

A **cocktail party graph** $\text{CP}(n)$ is the graph with $2n$ vertices $a_1, \ldots, a_n, b_1, \ldots, b_n$ in which $a_i$ is joined to every vertex except $b_i$ (and the same with $a$ and $b$ reversed).

Given a labelling $l$ of the vertices of $\Gamma$ with non-negative integers, the corresponding **generalized line graph** is the union of $L(\Gamma)$ with cocktail party graphs $\text{CP}(l(v))$ for all vertices $v$ of $\Gamma$, where an edge $\{v,w\}$ is joined to the cocktail party graphs corresponding to $v$ and $w$.

The next slide gives an example.

A generalized line graph

The figure shows a graph $\Gamma$ and the generalized line graph $L(\Gamma; (2, 1, 0, 3))$. Are these two graphs “the same”? More mathematically, are they isomorphic, in the sense that there is a bijection from the vertex set of the first to that of the second which maps edges to edges and non-edges to non-edges? Try it. You will fairly quickly construct an isomorphism.
<table>
<thead>
<tr>
<th>The theorem</th>
<th>Applications</th>
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<tbody>
<tr>
<td><strong>Theorem</strong></td>
<td>The theorem has various applications in graph theory. For example, it is not hard to show that, if a generalised line graph is regular, then it must be a line graph or a cocktail party graph; so a regular graph with least eigenvalue $-2$ must be of one of these types. There are applications further afield. Peter Sarnak, who was this year’s London Mathematical Society Hardy Lecturer, told us about an application to an engineering problem involving the design of waveguides.</td>
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<td>A connected graph with least eigenvalue $-2$ or greater is either a generalised line graph or one of a finite number of other graphs.</td>
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<td>The theorem is proved using the concept of root systems from the theory of Lie algebras (though these arise in many other areas of mathematics). The Petersen graph has smallest eigenvalue $-2$ and is one of the finite list of exceptions. In fact, the exceptional graphs are all represented by subsets of the exceptional root system $E_8$ with all products non-negative, two vertices joined if their inner product is positive. The complete list has not been computed, but we know all the regular graphs with smallest eigenvalue $-2$ which are not line graphs. Unsurprisingly, the Petersen graph is one of these.</td>
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<td><strong>Cospectral graphs</strong></td>
<td><strong>Strongly regular graphs</strong></td>
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<td>The spectrum of the adjacency matrix does not determine the graph up to isomorphism. (If graphs with adjacency matrices $A$ and $B$ are cospectral then $B = PAP^{-1}$ for some orthogonal matrix $P$, to be isomorphic we require $P$ to be a permutation matrix.) This is why linear algebra doesn’t solve the graph isomorphism problem. Indeed there can be more than exponentially many graphs on $n$ vertices with the same spectrum, something which we now explore.</td>
<td>A graph $\Gamma$ is strongly regular with parameters $(n,k,\lambda,\mu)$ if</td>
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<td>▶ it has $n$ vertices;</td>
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<td>▶ every vertex has $k$ neighbours;</td>
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<td>Two strongly regular graphs are cospectral if and only if they have the same parameters.</td>
<td>Indeed the eigenvalues and their multiplicities can be calculated from the parameters. The fact that the multiplicities are non-negative integers puts constraints on the parameters, which are necessary conditions for the existence of strongly regular graphs.</td>
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<td><strong>Latin squares</strong></td>
<td><strong>Steiner triple systems</strong></td>
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<td>A Latin square of order $n$ is an $n \times n$ array with each cell containing an entry from an alphabet of size $n$, such that any row or any column contains each symbol in the alphabet exactly once. Let $V$ be the set of cells of an $n \times n$ Latin square. Form a graph with vertex set $V$ by joining two cells if they lie in the same row, or same column, or have the same entry in the Latin square. This Latin square graph is strongly regular, with parameters $(n^2,3(n-1),n)$. Moreover, the Latin square graph determines the Latin square uniquely (up to the appropriate notion of isomorphism of Latin squares). Now the number of non-isomorphic Latin squares is very roughly $(n/c)^n$, for some constant $c$. This is asymptotically bigger than $\exp(n^2)$, giving more than exponentially many cospectral graphs. Moreover, almost all of them have trivial automorphism group.</td>
<td>A Steiner triple system consists of a set $P$ of $n$ points and a collection $B$ of triples or 3-element subsets of $P$ with the property that any two points of $P$ lie in a unique triple. The corresponding Steiner graph has vertex set $B$, two vertices adjacent if they have non-empty intersection. The Steiner graph is strongly regular, with parameters $(n(n-1)/6,3(n-3)/2,(n+3)/2,9)$. Steiner triple systems exist if and only if $n$ is congruent to 1 or 3 (mod 6). As before, for admissible values of $n$, there are so many of them that we again obtain more than exponentially many cospectral graphs, almost all of which have trivial automorphism groups.</td>
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Most strongly regular graphs?

If we list strongly regular graphs by number of vertices, then no obvious pattern occurs. Things are different if we list them by smallest eigenvalue, because of a remarkable theorem of Arnold Neumaier. I will only state a special case; a version of the theorem is true for any prescribed smallest eigenvalue.

Theorem
Let \( \Gamma \) be a strongly regular graph with smallest eigenvalue \(-3\). Then either
- \( \Gamma \) is a complete multipartite graph with parts of size 3;
- \( \Gamma \) is a Latin square graph;
- \( \Gamma \) is a Steiner graph;
- \( \Gamma \) belongs to a finite list of exceptions.

The analogous regularity condition is as follows. \( \Gamma \) is \( t \)-tuple regular if, whenever two sets \( A, B \) of vertices of size at most \( t \) have isomorphic induced subgraphs, then the number of common neighbours of \( A \) is equal to the number of common neighbours of \( B \).

Thus, \( 1 \)-tuple regular means "regular", while \( 2 \)-tuple regular means "strongly regular".

The following theorem does not require CFSG: the proof is just linear algebra.

Theorem
Let \( \Gamma \) be a \( 5 \)-tuple regular graph. Then \( \Gamma \) is homogeneous, and is one of the following:
- a disjoint union of complete graphs of the same size;
- a complete multipartite graph with all parts of the same size;
- the \( 5 \)-cycle;
- the line graph of \( K_{3,3} \).

Regular symmetry

Being strongly regular is a very strong regularity condition on a graph, but as we have seen, it does not imply the existence of any symmetry. Can we strengthen the condition so as to obtain symmetry from regularity?

A graph \( \Gamma \) is \( t \)-homogeneous if any isomorphism between induced subgraphs of \( \Gamma \) on at most \( t \) vertices extends to an automorphism of \( \Gamma \). It is homogeneous if it is \( t \)-homogeneous for all \( t \leq n \), where \( n \) is the number of vertices. This is a very strong symmetry condition.

For example, \( 1 \)-homogeneous means "vertex-transitive", while \( 2 \)-homogeneous means "transitive on vertices, ordered edges, and ordered non-edges". In group theoretic terms, the automorphism group has rank \( 3 \), that is, three orbits on ordered pairs of vertices.

Using the Classification of Finite Simple Groups, it is possible to write down a list of all the \( 2 \)-homogeneous finite graphs.

Graphs and groups

In the group-theoretical side of algebraic graph theory, the main concern is graphs with some symmetry, such as vertex-transitive graphs. But there are other results available. Here is an example. It is known that almost all graphs have trivial automorphism group (in the sense that the proportion of \( n \)-vertex graphs which have a non-trivial automorphism tends rapidly to zero as \( n \to \infty \)). What if we condition on the graph having some fixed group of automorphisms?

Theorem
Let \( \Gamma \) be a finite group. Let \( a_n(G) \) be the number of \( n \)-vertex graphs \( \Gamma \) for which \( G \leq \text{Aut}(\Gamma) \), and \( b_n(G) \) the number for which equality holds. Then \( b_n(G)/a_n(G) \) tends to a limit as \( n \to \infty \).

The limit is not always one. Indeed, for nilpotent groups, values of the limit are dense in \([0, 1]\).

Vertex-transitive graphs

The graph \( \Gamma \) is vertex-transitive if, for any two vertices, there is an automorphism of \( \Gamma \) mapping one to the other. More generally, if \( G \) is a subgroup of \( \text{Aut}(\Gamma) \), then \( \Gamma \) is \( G \)-vertex-transitive if the automorphism in the definition can be chosen to lie in \( G \).

Vertex-transitive graphs are regular, but they form a proper subclass of the class of regular graphs, and indeed have some special properties, for example:

Theorem
A vertex-transitive graph on an even number of vertices has a perfect matching (a set of pairwise disjoint edges covering the vertex set).

To examine vertex-transitive graphs further, we need to look at permutation groups.

Permutation groups

Let \( G \) be a permutation group on the set \( \Omega \). (This means that \( G \) is a subgroup of the symmetric group, the group of all permutations of \( \Omega \) (with the group operation being composition of permutations). We say that \( G \) acts transitively on \( \Omega \) if, for all \( a, b \in \Omega \), there exists \( g \in G \) with \( ag = b \). The stabiliser in \( G \) of a point \( a \in \Omega \) is the set

\[ G_a = \{ g \in G : ag = a \}. \]

It is in fact a subgroup of \( \Omega \). Stabilisers of different points in a transitive group are conjugate subgroups.

We say that \( G \) acts regularly on \( \Omega \) if it acts transitively on \( \Omega \) and \( G_a = \{ 1 \} \) for some (and hence all) \( a \in \Omega \).
### Orbitals and orbital graphs

Let $G$ be transitive on $\Omega$. An **orbital of $G$** is an orbit of $G$ on $\Omega \times \Omega$, the set of ordered pairs of elements of $\Omega$. Thus there is a unique **diagonal orbital** $\{(a, a) : a \in \Omega\}$. $G$ acts 2-transitively if there is a unique non-diagonal orbital (that is, any two distinct elements can be mapped to any other such pair by an element of $G$).

A non-diagonal orbital $O$ is **self-paired** if $(a, b) \in O$ implies $(b, a) \in O$.

Given an orbital $O$, we define the **orbital graph** to have vertex set $\Omega$ and edge set $O$. It is a directed graph if and only if $O$ is not self-paired. (We think of an undirected edge as the union of two oppositely-directed edges.)

Each orbital graph admits $G$ as an **arc-transitive** group of automorphisms. If $O$ is not self-paired and we define the graph with edge set the union of $O$ and the paired orbital, then we obtain an **undirected orbital graph**, on which $G$ acts edge-transitively.

### Cayley graphs

If $G$ acts regularly on $\Omega$, then we can identify $\Omega$ with $G$, where an arbitrary element $a$ of $\Omega$ is identified with the identity, and then $\beta = ag$ is identified with $g \in G$.

With this identification, the action of $G$ on itself is by **right multiplication**. The orbitals are the sets $O_s = \{(x, sx) : x \in G\}$ for each $s \in G$; the orbital $O_1$ is paired with $O_1$. Thus a $G$-invariant undirected graph on $G$ has edge set $\{(x, sx) : x \in G, s \in S\}$, where $S$ is an inverse-closed subset of $G \setminus \{1\}$. Such a graph is called a **Cayley graph** of $G$.

**Theorem**

$\Gamma$ is a Cayley graph for a group $G$ if and only if $\text{Aut}(\Gamma)$ contains a subgroup acting regularly on the vertex set of $\Gamma$.

In particular, Cayley graphs form an important subclass of vertex-transitive graphs.

### Vertex-transitive graphs and Cayley graphs

The converse is false: not every vertex-transitive graph is a Cayley graph.

For example, consider the Petersen graph. Its automorphism group is isomorphic to the symmetric group $S_5$; the only subgroup of order 10 in $S_5$ is the dihedral group, which contains involutions. But the vertices of the Petersen graph are identified with 2-sets from the set on which $S_5$ acts; and an involution fixes the 2-sets corresponding to its cycles of length 2.

Investigations by Brendan McKay, Cheryl Praeger and others seem to show that “most” vertex-transitive graphs are Cayley graphs, but as yet this is unproven. However, Gerd Sabidussi showed that every vertex-transitive graph has a cover (in a suitable sense) which is a Cayley graph.

### A warning

In the literature you will meet the concept of a normal Cayley graph. Unfortunately, the term has two incompatible meanings, both of which are very interesting.

- Some people call a Cayley graph $\text{Cay}(G, S)$ normal if the connection set $S$ is a normal subset of $G$ (closed under conjugation). Equivalently, the graph admits both left and right translations by $G$ as automorphisms.
- A more recent usage which has become common is that a Cayley graph for $G$ is normal if the right translations by elements of $G$ form a normal subgroup of the full automorphism group of the graph.

I’d like to change the terminology. Maybe we could call the first type **inner-automorphic** since it is preserved by inner automorphisms of $G$?

### All $G$-invariant graphs

**Proposition**

Let $G$ be a transitive permutation group on $\Omega$. Then any graph on the vertex set $\Omega$ which is $G$-invariant has edge set the union of some self-paired orbitals and some pairs of paired orbitals for $G$.

A similar result describes all the $G$-invariant directed graphs: the edge sets are arbitrary unions of orbitals.

### Graphs on groups

I will finish with a topic which has recently seen a lot of interest: concerns graphs whose vertex set is a group, but unlike (most) Cayley graphs it reflects the structure of the group in some way, and it is invariant under all automorphisms of the group.

Examples include:

- the **commuting graph**: $x$ and $y$ are joined if $xy = yx$.
- the **power graph**: $x$ and $y$ are joined if one of them is a power of the other.
- the **generating graph**: $x$ and $y$ are joined if the group $\langle x, y \rangle$ they generate is the whole of $G$.

The commuting graph was defined in the seminal paper of Brauer and Fowler in 1955, arguably the first step in the long journey to the Classification of Finite Simple Groups.

For a survey with more details, see my forthcoming survey in the *International Journal of Group Theory*. 
A hierarchy

The graphs in the following list form a hierarchy, in the sense that the edge set of each is contained in that of the next. (For the penultimate step we require that the group is not 2-generated abelian.)

- The null graph.
- The power graph.
- The enhanced power graph: $x$ and $y$ are joined if they are both powers of an element $z$ (in other words, $(x, y)$ is cyclic).
- The commuting graph.
- The non-generating graph (the complement of the generating graph).
- The complete graph.

Directed power graph

There is also a directed power graph, with an arc $x \rightarrow y$ if $y$ is a power of $x$. This relation is reflexive (if we add loops at each vertex) and transitive, that is, a partial preorder, and the power graph is its comparability graph. Using this we can show that the power graph is the comparability graph of a partial order, and hence (by Dilworth’s Theorem) it is perfect (that is, any induced subgraph has clique number equal to chromatic number).

Theorem

For two groups $G_1$ and $G_2$, the following are equivalent:

- their power graphs are isomorphic;
- their enhanced power graphs are isomorphic;
- their directed power graphs are isomorphic.

We do not have a good characterization of pairs of groups for which these conditions hold.

Isoclinism

The notion of isoclinism of groups was introduced by Philip Hall. Roughly it says that the commutation structure of two groups is the same. Commutation in a group $G$ can be regarded as a map $\gamma : G/Z(G) \times G/Z(G) \rightarrow G^\prime$, where $Z(G)$ and $G^\prime$ are the centre and derived subgroup of $G$. We say that $G_1$ and $G_2$ (with commutation maps $\gamma_1$ and $\gamma_2$) are isoclinic if there are isomorphisms $a : G_1/Z(G_1) \rightarrow G_2/Z(G_2)$ and $\beta : G_1^\prime \rightarrow G_2^\prime$ such that $(a, a) \gamma_2 = \gamma_1 \beta$.

Theorem

If $G_1$ and $G_2$ are isoclinic groups of the same order, then their commuting graphs are isomorphic.

The converse is known in some cases: simple groups, abelian groups, extraspecial groups, . . . . Indeed, I know no examples where it fails.

Universality

We saw that the power graph is the comparability graph of a partial order. In terms of induced subgraphs, this is the only restriction; and there is no comparable restriction for the other graphs in the hierarchy:

Theorem

Let $\Gamma$ be the comparability graph of a partial order. Then there is a finite group $G$ such that $\Gamma$ is an induced subgraph of the power graph of $G$.

Let $\Gamma$ be any finite graph. Then, for any of the other graph types in the hierarchy, there is a group $G$ such that $\Gamma$ is an induced subgraph of the graph of that type defined on the group $G$.

Some questions remain. For example, what is the smallest group required to embed a given graph, or to embed all graphs of a given order?

Equality?

When can consecutive graphs in the hierarchy be equal? This leads to some interesting group-theoretic questions:

- The power graph is null if and only if $G$ is the trivial group (for the identity is joined to all other vertices).
- The enhanced power graph is equal to the power graph if and only if $G$ contains no subgroup $C_p \times C_q$ where $p$ and $q$ are distinct primes; equivalently, every element has prime power order. These were first investigated by Higman in 1956, and have recently been classified.
- The commuting graph is equal to the enhanced power graph if and only if $G$ has no subgroup $C_p \times C_q$ for $p$ prime. These are the groups whose Sylow subgroups are cyclic or generalized quaternion; again they have all been classified.

- For $G$ not 2-generated abelian, the commuting graph is equal to the non-generating graph if and only if $G$ is a minimal non-abelian group. These groups were all determined by Miller and Moreno in 1904.
- The non-generating graph is complete if and only if $G$ is not 2-generated.

If two graphs in the hierarchy are not equal, we can ask about the graph whose edge set is the difference of their edge sets. In the extreme cases, the difference between the power graph and the null graph is the power graph, while the difference between the complete graph and the non-generating graph is the generating graph; both of these have been extensively studied. Little else has been done apart from Saul Freedman’s work on the difference between the non-generating graph and the commuting graph.
The power graph

I cannot summarise all that is known, but will give a few recent results about the graphs in the hierarchy.
In the power graph of a group $G$, every edge (and hence every clique) is contained in a cyclic subgroup of $G$. So the clique number is the maximum clique number of the cyclic subgroups of $G$.
Define the function $f$ by the rule that $f(n)$ is the clique number of the power graph of $C_n$. Then $f$ is given by the recurrence
- $f(1) = 1$;
- $f(n) = \phi(n) + f(n/p)$, where $\phi$ is Euler’s totient and $p$ the smallest prime divisor of $n$.

Using this, one can show that $\phi(n) \leq f(n) \leq 3\phi(n)$. In fact, 
$$\limsup \frac{f(n)}{\phi(n)} = 2.6481017597 \ldots$$
the constant on the right is
$$c = \sum_{k=0}^{\infty} \prod_{n=1}^{\infty} \frac{1}{p_n - 1},$$
where $p_1, p_2, \ldots$ are the primes in order. This constant is a limit point of values of $f(n)/\phi(n)$. Can all the limit points be described?

The enhanced power graph

We know that the power graph is a spanning subgraph of the enhanced power graph, with equality if and only if $G$ is an EPPO group (all elements of prime power order).
So, if $p$ is a monotone graph parameter, then $p(\text{Pow}(G)) \leq p(\text{EPow}(G))$. Asking when equality holds is a generalisation of the problem of determining the EPPO groups.
Here is one parameter for which there is a surprising answer:

**Theorem**
For any finite group $G$, the matching numbers of the power graph and the enhanced power graph of $G$ are equal.
However, we do not have a general formula for the matching number!

The commuting graph

The commuting graph is always connected, since the elements of the centre are joined to all other vertices. So it is common to study the reduced commuting graph, obtained by deleting the centre.
The following striking result is due to Michael Giudici, Luke Morgan and Chris Parker:

**Theorem**
- There is no upper bound for the diameter of the reduced commuting graph of a finite group; for any given $d$ there is a 2-group whose reduced commuting graph is connected with diameter greater than $d$.
- Suppose that the finite group $G$ has trivial centre. Then every connected component of its reduced commuting graph has diameter at most 10.

The generating graph

The following theorem of Tim Burness, Bob Guralnick and Scott Harper settles an old conjecture. Let $\Gamma$ be the generating graph of $G$ with the identity vertex removed.

**Theorem**
For a finite group $G$, the following conditions are equivalent:
- every vertex of $\Gamma$ has a neighbour (so there are no isolated vertices);
- every two vertices of $\Gamma$ have a common neighbour (so the diameter is at most 2);
- every proper quotient of $G$ is cyclic.
In particular, these conditions hold if $G$ is a non-abelian finite simple group.

The reference for the last part of my talk is
- Peter J. Cameron, Graphs defined on groups, *International J. Group Theory*, in press; https://ijgt.ui.ac.ir/article_25608_41a80b7b7cd84f2f2ba524c3e1d7a050.pdf

... for your attention.