Synchronizing automata, de Bruijn graphs, and applications

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Reset words are useful to bring a machine into a known state before applying further transformations to it.
An infamous problem

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![Automaton diagram]

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Show that, if an $n$-state automaton is synchronizing, it has a reset word of length at most $(n - 1)^2$. 

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For such a word reduces by (at least) one the number of reachable states. So after at most $n - 1$ such words we arrive at a single state. Now the next slide shows how this can be tested.
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The resulting word has length $O(n^2)$, giving an $O(n^3)$ upper bound for the length of a reset word. The constant has been improved, but not the exponent 3.
The Černý conjecture seems to have nothing to do with either graphs or algebraic structures; but there are connections, as we will see. Each letter of the alphabet corresponds to a transition on the set $\Omega$ of states. Reading a word corresponds to composing the transitions. So the set of all possible transitions is closed under composition and contains the identity map (corresponding to the empty word): so

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So the Černý conjecture is a question about transformation monoids, and semigroups enter the picture.
Graphs

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The endomorphisms of a graph form a transformation monoid. Moreover, as long as the graph has at least one edge, its endomorphism monoid is not synchronizing, since that edge cannot be collapsed by any endomorphism.
Synchronization and endomorphisms

Now we have a pleasant surprise:

Theorem

A transformation monoid $M$ is non-synchronizing if and only if there is a non-trivial graph $\Gamma$ on the domain such that $M$ is contained in the endomorphism monoid of $\Gamma$. Moreover, we can assume that the clique number and chromatic number of $\Gamma$ are equal.

A graph is trivial if it is complete (all possible edges) or null (no edges at all). The clique number is the number of vertices in the largest complete subgraph, while the chromatic number is the number of colours required to colour the vertices so that adjacent vertices get different colours.

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For the converse, let $M$ be a transformation monoid on $\Omega$. We define a graph $\text{Gr}(M)$ as follows: the vertex set is $\Omega$; there is an edge joining $s$ and $t$ if and only if there is no element $m \in M$ with $sm = tm$. Now
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The first point is clear; I will outline the second. If it fails, then some element $m \in M$ maps an edge $\{s, t\}$ to either a single vertex or a non-edge. The first case contradicts the definition; in the second case, there is $m' \in M$ with $(sm)m' = (tm)m'$, so $mm'$ maps $s$ and $t$ to the same place.
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For the last point, take an element \( m \in M \) of minimal rank; then \( m \) is a colouring of the graph and its image is a clique.
Does this help?

We seem to have replaced an easy problem (deciding whether an automaton is synchronizing) by a much harder problem (deciding whether the graph has clique number equal to chromatic number).
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Strong synchronization

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This condition, as we will see, is closely connected with automorphisms of the shift map in symbolic dynamics.
De Bruijn graphs

Let $n$ be a positive integer and $A$ a finite alphabet. The de Bruijn graph $G(n, A)$ has vertex set $A^n$. For $a \in A$, $w \in A^n$, the target of the edge labelled $a$ with source $w$ is obtained by removing the first letter of $w$ and appending $a$. 

![Graph](image-url)
De Bruijn graphs

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The de Bruijn graph as automaton

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It seems clear that it is in some sense the “universal” automaton which is strongly synchronizing at level $n$. We now turn to this.
A **folding** of an automaton is an equivalence relation $\equiv$ on the set of states having the property that, if states $s$ and $t$ are equivalent, and $s'$ and $t'$ are the states resulting from reading a given letter $a$ from these two states, then $s'$ and $t'$ are equivalent.
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Foldings

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**Problem**

*If $|A| = k$, how many foldings of $G(n, A)$ are there?*
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Richard Brautigan, *The Hawkline Monster: A Gothic Western*
Counting foldings

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*How many foldings of the de Bruijn graph with word length $n$ over an alphabet of size $q$ are there?*
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The formula for $n = 2$ is messy to state, but easy to compute: the numbers of foldings for $|A| = 2, \ldots, 7$ are 5, 192, 78721, 519338423, 82833228599906, 429768478195109381814.
Thompson’s groups

Three of the best-studied infinite groups were discovered by Richard Thompson in the 1950s, and are known as $F$, $T$ and $V$. Here are brief descriptions.
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![Graph showing piecewise-linear order-preserving permutations](image-url)
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![Graph of the unit interval with dyadic rationals](image)

Representing numbers in the unit interval by dyadic rationals, we see that the group acts by **prefix replacement**: in the above example, $00x \rightarrow 0x$, $01x \rightarrow 10x$, $1x \rightarrow 11x$. 
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In product replacement form this is $00x \mapsto 1x$, $01x \mapsto 010x$, $10x \mapsto 011x$, and $11x \mapsto 00x$. 
The Higman–Thompson groups

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To relate these groups to the previous discussion, we introduce the notion of a transducer: this is an automaton which has the capacity to write as well as read symbols from an alphabet. In general, a transducer reads a symbol, changes state, and writes a string of symbols from the alphabet (possibly empty). In order to avoid trivial cases, we always assume that when a transducer reads an infinite string of symbols, it writes out an infinite string: in other words, if we traverse a cycle in the digraph of the underlying automaton, at least one symbol is written. As just hinted, a transducer $A$ with a prescribed starting state $s$ (called an initial transducer) can be regarded as defining a map from the set $A^\omega$ of infinite strings over the alphabet $A$ to itself. We are interested in the case where this map is invertible, and the inverse is also represented by a transducer.
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To relate these groups to the previous discussion, we introduce the notion of a **transducer**: this is an automaton which has the capacity to write as well as read symbols from an alphabet. In general, a transducer reads a symbol, changes state, and writes a string of symbols from the alphabet (possibly empty). In order to avoid trivial cases, we always assume that **when a transducer reads an infinite string of symbols, it writes out an infinite string**: in other words, if we traverse a cycle in the digraph of the underlying automaton, at least one symbol is written.
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The rational group $\mathcal{R}_n$ over an $n$-letter alphabet $A$ was defined by Grigorchuk, Nekrashevych, and Suschanskiĭ.
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The automorphism group of $G_{n,r}$

An invertible initial transducer is said to be **bisynchronizing** if the underlying automaton is strongly synchronizing, and the same holds for the automaton representing its inverse.

**Theorem**
The automorphism group of $G_{n,r}$ is the group of transformations of $A^\omega$ induced by bisynchronizing initial transducers; so it is a subgroup of the rational group $\mathcal{R}_{n,r}$.
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**Theorem**

The automorphism group of $G_{n,r}$ is the group of transformations of $A^\omega$ induced by bisynchronizing initial transducers; so it is a subgroup of the rational group $R_{n,r}$.

This theorem is proved in the paper of Bleak, Cameron, Maissel, Navas and Olukoya (arXiv 1605.09302).
Consequences

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**Theorem**

*The outer automorphism group of $G_{n,r}$ has trivial centre and unsolvable order problem.*

The proof involves a connection between $\text{Out}(G_{n,r})$ and the automorphism group of the two-sided shift in symbolic dynamics, allowing known results about the second to be transferred to the first. I turn now to this.
The **shift map** $\sigma$ comes in two flavours. It acts on either the set $A^\omega$ of infinite strings of symbols from $A$, or on the set $A^\mathbb{Z}$ of two-way infinite strings; it moves each symbol one place to the left. (In the one-way case, the first symbol of the string is lost, so the shift is onto but not one-to-one; in the second case it is a bijection.)
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The shift map is the central character in symbolic dynamics, arising from a discretisation of dynamics of (for example) planetary orbits.
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Two recent papers by Bleak, Cameron and Olukoya (arXiv 2004.08478 and 2006.01466) use transducers to study the automorphism groups of the shift maps. Some of the results are new; several give simpler proofs of known results, or versions more suitable to actual computation. Here are some examples.
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Two recent papers by Bleak, Cameron and Olukoya (arXiv 2004.08478 and 2006.01466) use transducers to study the automorphism groups of the shift maps. Some of the results are new; several give simpler proofs of known results, or versions more suitable to actual computation. Here are some examples. First, it is noted that the automorphism group of the one-sided shift over an $n$-letter alphabet embeds into the group of outer automorphisms of $G_{n,r}$: the automorphisms are given by bisynchronizing transducers.
In the one-sided case, the orders of torsion elements of $\text{Aut}(\sigma)$ are orders of automorphism groups of foldings of de Bruijn graphs.
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In the two-sided case, \( \text{Aut}(\sigma) \) contains the group generated by \( \sigma \) as a central subgroup; the quotient is embeddable in the group of outer automorphisms of \( G_{n,r} \).
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In the two-sided case, $\text{Aut}(\sigma)$ contains the group generated by $\sigma$ as a central subgroup; the quotient is embeddable in the group of outer automorphisms of $G_{n,r}$.

In this case, automorphisms are specified by an annotated transducer, where the transducer determines the coset of $\langle \sigma \rangle$, and the annotation determines the element of this coset.

Collin Bleak, Peter Cameron, Yonah Maissel, Andrés Navas, and Feyishayo Olukoya, The further chameleon groups of Richard Thompson and Graham Higman: Automorphisms via dynamics for the Higman groups $G_{n,r}$, arXiv 1605.09302.


... for your attention.