Regular polytopes

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This is joint work with Maria Elisa Fernandes (Aviero), Dimitri Leemans (Auckland) and Mark Mixer (Boston). This year I spent nearly two months in Auckland, thanks to support from the Hood Fellowship, where I learned about regular polytopes and carried out the research reported here.

Independent generating sets

Let \( G \) be a finite group. A set \( \{g_1, \ldots, g_r\} \) of elements of \( G \) is independent if none of the elements lies in the subgroup generated by the others. It is an independent generating set if, in addition, the whole set generates the group \( G \). Thus independent generating sets resemble bases for vector spaces in elementary linear algebra. However, they do not have the nice properties of bases such as the exchange property, and so they are not the bases of a matroid.

Polytopes

Polytopes are objects which have combinatorial, geometric and algebraic aspects. I will be particularly concerned with regular polytopes, which are generalisations of the classical regular polyhedra in 3-space. They are polytopes which have the maximal amount of symmetry (in a precise sense), and not surprisingly their study has very close connections with group theory.

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In the symmetric group

Theorem (Julius Whiston, 2000)

The largest size of an independent set in the symmetric group \( S_n \) is \( n - 1 \); equality holds if and only if the set is an independent generating set.

In 2002, Philippe Cara and I found all the independent generating sets of size \( n - 1 \) in \( S_n \), for \( n \geq 7 \). There are two types:

- The first type consists of the transpositions corresponding to the edges of a tree on \( n \) vertices.
- The second type contains one transposition; the other elements are 3-cycles and double transpositions. These will not be relevant in what follows.

There are a few extra types for small \( n \). For example, for \( n = 6 \), we can take images of the above types under the outer automorphism of \( S_6 \).

Subgroup lattices

Let \( L(G) \) denote the subgroup lattice of the group \( G \).

Proposition

For any finite group \( G \), the Boolean lattice \( B(r) \) is embeddable as a meet-semilattice of \( L(G) \) if and only if it is embeddable as a join-semilattice of \( L(G) \). The largest number \( r \) for which these equivalent properties hold is equal to the size of the largest independent subset of \( G \).

If \( \{g_1, \ldots, g_r\} \) is an independent set in \( G \), then the subgroups generated by its subsets form a join-semilattice of \( L(G) \) isomorphic to \( B(r) \).

Note that the above conditions are not equivalent to the embeddability of \( B(r) \) in \( L(G) \) as a lattice!

Polytopes

A polytope of dimension \( r \) is a generalisation of polygon (in 2 dimensions) or polyhedron (in 3 dimensions) to arbitrary dimension.

It can be regarded as a partially ordered set (the elements are the faces of various dimensions) in which all maximal chains contain \( r + 2 \) elements (including a bottom element \( \emptyset \) of dimension \( -1 \) and a top element of dimension \( r \) which represents the whole polytope). Each element can be assigned a unique dimension, corresponding to the position it occupies in a maximal flag. Elements of dimension 0, 1, 2 are vertices, edges, and faces.

The maximal chains are called flags.

We require several further conditions (see next slide).
Regular polytopes

If two flags \((x_{-1}, x_0, \ldots, x_i, x_{i+1}, \ldots, x_r)\) and
\((x_{-1}, y_0, \ldots, y_i, y_{i+1}, \ldots, y_r)\) differ only in the element of
dimension \(i\), then any automorphism which fixes the first flag
also fixes the second.

Hence, using the strong connectedness property, any
automorphism which fixes a flag must fix every flag, and hence
is the identity.

A polytope is regular if the automorphism group acts
transitively on the flags. In this situation, the action of
the group is regular: there is a bijection between flags and
automorphisms. (We fix a reference flag \(F\) and then identify
\(F\) with the unique automorphism mapping \(F\) to another.

If a polytope is regular, then, for any \(i\), if \(\dim(x) = i - 1\),
\(\dim(y) = i + 2\), and \(x < y\), then \([x, y]\) is a \(p_i\)-gon, where \(p_i\)
depends on \(i\) but not on \(x\) and \(y\). The vector \((p_0, p_1, \ldots, p_{r-1})\) is
the Schläfli symbol of the polytope.

String C-groups

Because of the correspondence between the set of flags and the
automorphism group \(G\) of a polytope, it is possible to translate
everything into the group. We will see that the existence of a
regular polytope is equivalent to a sequence of group elements
with certain properties.

To motivate this, consider the cube.

```
   e
  /  \
 f  e
 / \
 c
/  \
 f
```

Our reference flag is \((\emptyset, v, e, f, C)\) (where \(C\) denotes the cube).
Let \(s_v\) and \(s_e\) be the automorphisms mapping it to
\((\emptyset, v', e, f, C)\), \((\emptyset, v, e', f, C)\) and \((\emptyset, v, e, f', C)\) respectively.

More generally, we define a string C-group to be a finite group
generated by elements \(s_0, s_1, \ldots, s_{r-1}\) satisfying the conditions

- \(s_i^2 = 1\).
- \(s_i\) and \(s_j\) commute if \(|i - j| > 1\) (the string condition).
- For \(I \subseteq \{0, \ldots, r-1\}\), let \(S_I\) denote the subgroup generated
by \(\{s_i : i \in I\}\). Then \(S_I \cap S_J = S_{I \cap J}\) for any \(I\) and \(J\) (the
intersection condition).

Theorem

The existence of a regular polytope with automorphism group \(G\) is
"equivalent" (in a suitable sense) to a representation of \(G\) as a string
C-group.

Note that the order of \(s_i s_{i+1}\) is the \(i\)th component of the Schläfli
symbol of the polytope.

We do not insist that \(s_i\) and \(s_j\) fail to commute if \(|i - j| > 1\).
In other words, we allow degenerate polytopes where some of the
polygons are digons. This might seem to make things harder,
but actually makes them much easier. The subgroup generated
by a subset of \(\{s_0, \ldots, s_{r-1}\}\) is a string C-group in its own right,
so we have the possibility of induction!

Also, we do not assume that the orders of the \(s_i\) and \(s_j s_i\) give a
presentation of a group. (If they do, then the group is a Coxeter
group.)

Finally, the intersection condition shows that \(\{s_0, \ldots, s_{r-1}\}\) is an
independent generating set for \(G\). Indeed, it is stronger: it is
equivalent to the condition that the map \(I \to G_I\) embeds the
Boolean lattice \(B(r)\) as a sublattice of the subgroup lattice \(L(G)\)
of \(G\).
The symmetric group, 1

It follows from Whiston’s theorem that the dimension of a polytope with automorphism group $S_n$ is at most $n - 1$. It further follows from the theorem of Cameron and Cara that there is a unique such polytope of rank $n - 1$. (The condition that generators are involutions rules out the second type; the string condition shows that the tree is a string.) The generators are $s_i = (i + 1, i + 2)$ for $i = 0, \ldots, n - 2$. The corresponding polytope is the regular $(n - 1)$-simplex, whose faces are all the subsets of $\{1, \ldots, n\}$.

The symmetric group, 2

Fernandes, Leemans and Mixer asked about regular polytopes of smaller dimension $r$ with group $S_n$. They computed the following table:

<table>
<thead>
<tr>
<th>$n \backslash r$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>35</td>
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<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>68</td>
<td>36</td>
<td>11</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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</tr>
<tr>
<td>9</td>
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<td>7</td>
<td>7</td>
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</tr>
<tr>
<td>10</td>
<td>413</td>
<td>203</td>
<td>52</td>
<td>13</td>
<td>7</td>
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<td>1</td>
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<td>0</td>
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</tr>
<tr>
<td>11</td>
<td>1221</td>
<td>189</td>
<td>43</td>
<td>25</td>
<td>9</td>
<td>7</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
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</tr>
<tr>
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<tr>
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<td>35</td>
<td>9</td>
<td>7</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

The alternating groups

We saw that regular polytopes with a given group (like $S_n$) can be studied by induction, using the fact that any subset of the generators of the string C-group themselves generate a smaller string C-group. Fernandes, Leemans and Mixer examined the alternating group $A_n$. They conjectured that the largest dimension of a regular polytope with group $A_n$ is $n/2 + 1$, with equality only for $n \equiv 2 \mod 4$. They managed to construct examples meeting the conjectured bound. This, incidentally, shows that there is a big difference between largest dimension of a polytope with group $G$, and largest independent generating set for $G$ (which is $n - 2$ for $G = A_n$).

Other subgroups of $S_n$

My contribution to this problem, after working intermittently with Dimitri Leemans on this, was the following theorem. ("Number" refers to the list of transitive groups of this degree in Magma.)

Theorem

A regular polytope of rank $r$ whose group $G$ is isomorphic to a transitive subgroup of $S_n$ other than $S_n$ or $A_n$ satisfies one of the following:

- $r \leq n/2$.
- $n \equiv 2 \mod 4$, $r = n/2 + 1$ and $G$ is $C_2 \wr S_n/2$. The generators are explicitly known; the Schläfli type is $(2, 3, \ldots, 3, 4)$.
- $G$ is transitive imprimitive and is one of the examples appearing in the table below.
- $G$ is primitive. In this case, $G$ is obtained from the permutation representation of degree 6 of $S_3 \cong \text{PGL}_2(5)$ and the polytope is the 4-simplex of Schläfli type $[3, 3, 3]$.

<table>
<thead>
<tr>
<th>Degree</th>
<th>Number</th>
<th>Structure</th>
<th>Order</th>
<th>Schläfli Type</th>
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</thead>
<tbody>
<tr>
<td>6</td>
<td>9</td>
<td>$S_3 \times S_3$</td>
<td>36</td>
<td>$[2, 3, 3]$</td>
</tr>
<tr>
<td>6</td>
<td>11</td>
<td>$2^2 : S_3$</td>
<td>48</td>
<td>$[2, 3, 3]$</td>
</tr>
<tr>
<td>6</td>
<td>11</td>
<td>$2^3 : S_3$</td>
<td>48</td>
<td>$[2, 3, 4]$</td>
</tr>
<tr>
<td>8</td>
<td>45</td>
<td>$2^4 : S_3 \times S_3$</td>
<td>576</td>
<td>$[3, 4, 4, 3]$</td>
</tr>
</tbody>
</table>

We hope to be able to use this result to prove the conjectured bound for the dimension of a polytope admitting the alternating group. It may also be of use in tackling the mysterious conjecture for the symmetric group.