Equitable partitions of Latin square graphs

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Equitable partitions

We have a graph $\Gamma$ on the vertex set $\Omega$; we assume that $\Gamma$ is connected and is regular with valency $k$. A partition $\Delta = \{\Delta_1, \ldots, \Delta_v\}$ of $\Omega$ is equitable if there is a matrix $M = (m_{ij})$ such that a vertex in $\Delta_i$ has exactly $m_{ij}$ neighbours in $\Delta_j$.

Examples:

- The orbits of a group of automorphisms of $\Gamma$.
- The distance partition with respect to any vertex is equitable with the same matrix if and only if the graph is distance-regular.
- Many examples in finite geometry, including ovoids, spreads, and Cameron–Liebler line classes, fit into this framework.

The spectrum

Let $\Gamma$ have adjacency matrix $A$. Let $\Delta$ be an equitable partition with matrix $M$. If $v_i$ is the characteristic function of $\Delta_i$, then

$$v_iA = \sum v_im_{ij},$$

so the spectrum of $M$ is contained in that of $A$.

Let $A$ have eigenvalue $k$, the principal eigenvalue, with multiplicity 1; $n-3$ (with multiplicity $3(n-1)$), and $-3$ (with multiplicity $(n-1)(n-2)$).

Perfect sets

A subset $S$ of $\Omega$ is perfect if the partition $\{S, \Omega \setminus S\}$ is equitable; it is $\mu$-perfect if the partition is $\mu$-equitable.

Now easy linear algebra shows that a partition $\Delta$ is $\mu$-equitable if and only if all but at most one part of the partition is $\mu$-perfect.

In particular, to find all $\mu$-equitable partitions, it suffices to find all the minimal $\mu$-perfect sets.

Latin square graphs

A Latin square of order $n$ is an $n \times n$ array with entries from an alphabet of $n$ letters, such that each letter occurs once in each row and once in each column.

Given a Latin square $L$, we define the corresponding Latin square graph $\Gamma(L)$ to have as vertices the $n^2$ cells of the array $L$, two vertices joined if they lie in the same row or the same column or contain the same letter.

The eigenvalues of the adjacency matrix are $3(n-1)$ (the principal eigenvalue, with multiplicity 1); $n-3$ (with multiplicity $3(n-1)$), and $-3$ (with multiplicity $(n-1)(n-2)$).

First examples

Let $S$ be the set of $n$ cells in a row. Then $\{S, \Omega \setminus S\}$ is equitable, with matrix

$$\begin{pmatrix} n-1 & 2(n-1) \\ 2 & 3n-5 \end{pmatrix},$$

so $S$ is $(n-3)$-perfect. Of course, the same applies to any column or letter.

What G and G did

At the International Workshop on Bannai–Ito Theory in Hangzhou, Sergey Goryainov talked about a result he had proved with his supervisor Alexander Gavrilyuk. Although phrased in terms of bilinear forms, it amounted to a complete determination of the $(n-3)$-equitable partitions (or, equivalently, the minimal $(n-3)$-perfect sets) in a particular type of Latin square graph: the Cayley table of an elementary abelian 2-group.

The result is that these are rows, columns, letters, or one more type: subsquares of order $n/2$ corresponding to subgroups of index 2 in the group.

RAB and PJC wondered whether this could be generalised …
More examples

They found two new constructions of \((n - 3)\)-perfect sets:

- **Corner sets** in the Cayley tables of cyclic groups. These have shape

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 0 \\
3 & 4 & 0 & 1 \\
4 & 0 & 1 & 2 \\
\end{array}
\]

- **Inflation** Take a Latin square \(L_0\) of order \(s\). Replace each occurrence of letter \(i\) be a Latin square of order \(t\) in alphabet \(A_i\), where the alphabets for different letters are pairwise disjoint; this gives a Latin square \(L\) of order \(n = st\). Moreover, given an \((s - 3)\)-perfect set \(S_0\) in \(L_0\), the corresponding cells in \(L\) form an \((n - 3)\)-perfect set.

For example, inflating a single entry in the 2 \(\times\) 2 Latin square gives the G-G example.

The theorem

**Theorem**

Let \(S\) be a minimal \((n - 3)\)-perfect set in the graph of a Latin square of order \(n\). Then \(S\) is a row, a column, a letter, or an inflation of a corner set.

So we need no assumption about the structure of the Latin square. The proof is quite complicated and I have no time to describe it here.