Equitable partitions of Latin square graphs

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Equitable partitions

We have a graph $\Gamma$ on the vertex set $\Omega$. We assume that $\Gamma$ is connected and is regular with valency $k$. A partition $\{\Delta_1, \ldots, \Delta_r\}$ of $\Omega$ is equitable if there is a matrix $M = (m_{ij})$ such that a vertex in $\Delta_i$ has exactly $m_{ij}$ neighbours in $\Delta_j$.

In the Petersen graph, the partition of the vertices into inner and outer 5-cycle is equitable, since a vertex in one has a unique neighbour in the other.
Examples

- The orbits of a group of automorphisms of $\Gamma$. For two vertices in the same part are equivalent under an automorphism fixing all the parts.

- The distance partition with respect to any vertex $v$ (the partition into the sets of vertices at distance 0, 1, 2, \ldots from $v$) is equitable if and only if the graph is distance-regular. The condition is equivalent to the usual definition.
In finite geometry

Many examples in finite geometry can be thought of as equitable partitions. Among these are

- sets of disjoint Steiner systems $S(t, t + 1, n)$
- ovoids
- spreads
- hemisystems
- Cameron–Liebler line classes
- Many others!
The spectrum

Let $Γ$ have adjacency matrix $A$. Then $A$ is a real symmetric matrix, and so is diagonalisable by an orthogonal matrix; we refer to the spectrum of this matrix as the spectrum of $Γ$. Let $Δ$ be an equitable partition with matrix $M$. If $v_i$ is the characteristic function of $Δ_i$, then

$$v_jA = \sum v_im_{ij}.$$ 

So the subspace spanned by the vectors $v_i$ is invariant under $A$, and the restriction of $A$ to this space is just given by the matrix $M$ of the equitable partition. It follows that the spectrum of $M$ (which we will refer to as the spectrum of the equitable partition) is contained in the spectrum of the graph $Γ$. In the case of the distance partition of a distance-regular graph, the spectrum of $M$ has all the eigenvalues of $A$, each with multiplicity 1.
$M$ always has eigenvalue $k$, the principal eigenvalue, since its row sums are equal to $k$.
Let $\mu$ be an eigenvalue of $G$ different from $k$. We say that the partition is $\mu$-equitable if all non-principal eigenvalues of $M$ are equal to $\mu$. This means that the vectors $v_i$ all lie in the sum of the $k$- and $\mu$-eigenspaces of $A$.
Any two-part equitable partition is $\mu$-equitable for some $\mu$. But we will see that, for partitions with more than two parts, $\mu$-equitable partitions are easier to deal with than arbitrary equitable partitions.
Perfect sets

A subset $S$ of $\Omega$ is **perfect** if the partition $\{S, \Omega \setminus S\}$ is equitable; it is **$\mu$-perfect** if the partition is $\mu$-equitable. Now easy linear algebra shows that a partition $\Delta$ is $\mu$-equitable if and only if all but at most one part of the partition is $\mu$-perfect. In particular, to find all $\mu$-equitable partitions, it suffices to find all the **minimal** $\mu$-perfect sets.
Latin square graphs

A **Latin square** of order $n$ is an $n \times n$ array with entries from an alphabet of $n$ letters, such that each letter occurs once in each row and once in each column. Given a Latin square $L$, we define the corresponding **Latin square graph** $\Gamma(L)$ to have as vertices the $n^2$ cells of the array $L$, two vertices joined if they lie in the same row or the same column or contain the same letter.

- There are $n^2$ vertices.
- The valency is $3(n - 1)$: any cell is in the same row as $n - 1$ others, in the same column as $n - 1$ others, and contains the same letter as $n - 1$ others.
- If two cells are joined, they have $n - 2 + 1 + 1 = n$ common neighbours.
- If two cells are not joined, they have $2 + 2 + 2 = 6$ common neighbours.
The spectrum

Thus the graph is strongly regular; its adjacency matrix $A$ satisfies

$$A^2 = 3(n - 1) + nA + 6(J - I - A),$$

where $J$ is the all-1 matrix.

The eigenvalues of the adjacency matrix are $3(n - 1)$ (the principal eigenvalue, with multiplicity 1); $n - 3$ (with multiplicity $3(n - 1)$), and $-3$ (with multiplicity $(n - 1)(n - 2)$).
Let $S$ be the set of $n$ cells in a row. Then $\{S, \Omega \setminus S\}$ is equitable, with matrix

$$
\begin{pmatrix}
 n - 1 & 2(n - 1) \\
 2 & 3n - 5
\end{pmatrix}.
$$

Since the row sum is $3(n - 1)$ and the trace is $4n - 6$, the other eigenvalue is $n - 3$. So $S$ is $(n - 3)$-perfect.

Of course, the same applies to any column or letter.
What G and G did

At the International Workshop on Bannai–Ito Theory in Hangzhou, Sergey Goryainov talked about a result he had proved with his supervisor Alexander Gavrilyuk. Although phrased in terms of bilinear forms, it amounted to a complete determination of the \((n - 3)\)-equitable partitions (or, equivalently, the minimal \((n - 3)\)-perfect sets) in a particular type of Latin square graph: the Cayley table of an elementary abelian 2-group.

The result is that these are rows, columns, letters, or one more type: subsquares of order \(n/2\) corresponding to subgroups of index 2 in the group. See next slide.
RAB and PJC wondered whether this could be generalised. Certainly any Latin square of order $n$ with a subsquare of order $n/2$ will give an example. Is that all?
Inflation

Take a Latin square $L_0$ of order $s$. Replace each occurrence of letter $i$ by a Latin square of order $t$ in alphabet $A_i$, where the alphabets for different letters are pairwise disjoint; this gives a Latin square $L$ of order $n = st$. Moreover, given an $(s - 3)$-perfect set $S_0$ in $L_0$, the corresponding cells in $L$ form an $(n - 3)$-perfect set.

For example, inflating the Latin square

\[
\begin{array}{cc}
0 & 1 \\
1 & 0 \\
\end{array}
\]

using the same Latin square in each place (the Cayley table of an elementary abelian 2-group) gives the example of Gavrilyuk and Goryainov.
A corner set in the Cayley table of a cyclic group has shape

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Some analysis shows that it is an \((n - 3)\)-perfect set. A corner set in the order-2 Latin square is a single entry. So this extends even further the earlier examples.
The theorem

Theorem
Let $S$ be a minimal $(n - 3)$-perfect set in the graph of a Latin square of order $n$. Then $S$ is a row, a column, a letter, or an inflation of a corner set.

Let $S$ be $(n - 3)$-perfect. We can assume that it contains no row, column, or letter. A slice of $S$ is its intersection with a row, column or letter.
Choose a slice of maximum size, which (without loss) is the intersection of $S$ with a row. There is a set of $s$ columns meeting this row in a cell not in $S$. Then show that there are $s$ rows defining slices of the same size not meeting this set of columns; these $s$ rows and $s$ columns form a subsquare; and the $s$ columns are all disjoint from $S$.
This describes the “top right” of the given square. We use induction to work our way down and to the left, and end up showing that $S$ is an inflation of a corner set.
The other non-principal eigenvalue of a Latin square graph is $-3$. Can we say anything about $-3$-perfect sets? Such a set $S$ has the property that it meets any row, column or letter in a constant number $s$ of cells, and its cardinality is $sn$. In particular, with $s = 1$, such set is a transversal, a set containing one cell from each row, column or letter.

But not every $-3$-perfect set can be decomposed into transversals.
One of the oldest conjectures about Latin squares is Ryser’s conjecture, asserting that any Latin square of odd order has a transversal. Many squares of even order do too, but some do not (for example, the Cayley table of the cyclic group). The conjecture is still open despite a lot of work, so characterising such sets is unlikely to be achieved soon!
−3-equitable partitions

A special case of a −3-equitable partition would be a partition of the elements of the Latin square into transversals.

This is equivalent to the existence of an orthogonal mate to the given square:
Mixed equitable partitions

If we consider partitions where both non-principal eigenvalues occur, there is much more freedom, and probably no hope of a classification. To mention just two examples:

- The distance partition of the graph with respect to any cell. (The Latin square graph is distance-regular, so the non-principal eigenvalues each occur with multiplicity 1.)
- The $t \times t$ subsquares associated with a $t$-fold inflation of a Latin square of order $s$. (Both non-principal eigenvalues occur if and only if $s > 2$.)

And there are no doubt many more …
This morning (27 April), the paper was accepted for publication (after minor revision), just five months after we started the project.
See you at next year’s Scottish Combinatorics Meeting . . .